Spoon Feeding Differential Equations

Simplified Knowledge Management Classes Bangalore

My name is Subhashish Chattopadhyay. I have been teaching for IIT-JEE, Various International Exams (such as IMO [International Mathematics Olympiad], IPhO [International Physics Olympiad], IChO [International Chemistry Olympiad]), IGCSE (IB), CBSE, I.Sc, Indian State Board exams such as WB-Board, Karnataka PU-II etc since 1989. As I write this book in 2016, it is my 25 th year of teaching. I was a Visiting Professor to BARC Mankhurd, Chembur, Mumbai, Homi Bhabha Centre for Science Education (HBCSE) Physics Olympics camp BARC Campus.

I am Life Member of …

- IAPT (Indian Association of Physics Teachers)

- IPA (Indian Physics Association)

- AMTI (Association of Mathematics Teachers of India)

- National Human Rights Association

- Men's Rights Movement (India and International)

- MGTOW Movement (India and International)

And also of

IACT (Indian Association of Chemistry Teachers)

The selection for National Camp (for Official Science Olympiads - Physics, Chemistry, Biology, Astronomy) happens in the following steps ….

1) **NSEP** (National Standard Exam in Physics) and **NSEC** (National Standard Exam in Chemistry) held around 24 rth November. Approx 35,000 students appear for these exams every year. The exam fees is Rs 100 each. Since 1998 the IIT JEE toppers have been topping these exams and they get to know their rank / performance ahead of others.

2) **INPhO** (Indian National Physics Olympiad) and **INChO** (Indian National Chemistry Olympiad). Around 300 students in each subject are allowed to take these exams. Students coming from outside cities are paid fair from the Govt of India.

3) The Top 35 students of each subject are invited at HBCSE (Homi Bhabha Center for Science Education) Mankhurd, near Chembur, BARC, Mumbai. After a 2-3 weeks camp the top 5 are selected to represent India. The flight tickets and many other expenses are taken care by Govt of India.

Since last 50 years there has been no dearth of "Good Books". Those who are interested in studies have been always doing well. This e-Book does not intend to replace any standard text book. These topics are very old and already standardized.

There are 3 kinds of Text Books

- The thin Books - Good students who want more details are not happy with these. Average students who need more examples are not happy with these. Most students who want to "Cram" quickly and pass somehow find the thin books "good" as they have to read less !!

- The Thick Books - Most students do not like these, as they want to read as less as possible. Average students are "busy" with many other things and have no time to read all these.

- The Average sized Books - Good students do not get all details in any one book. Most bad students do not want to read books of "this much thickness" also !!

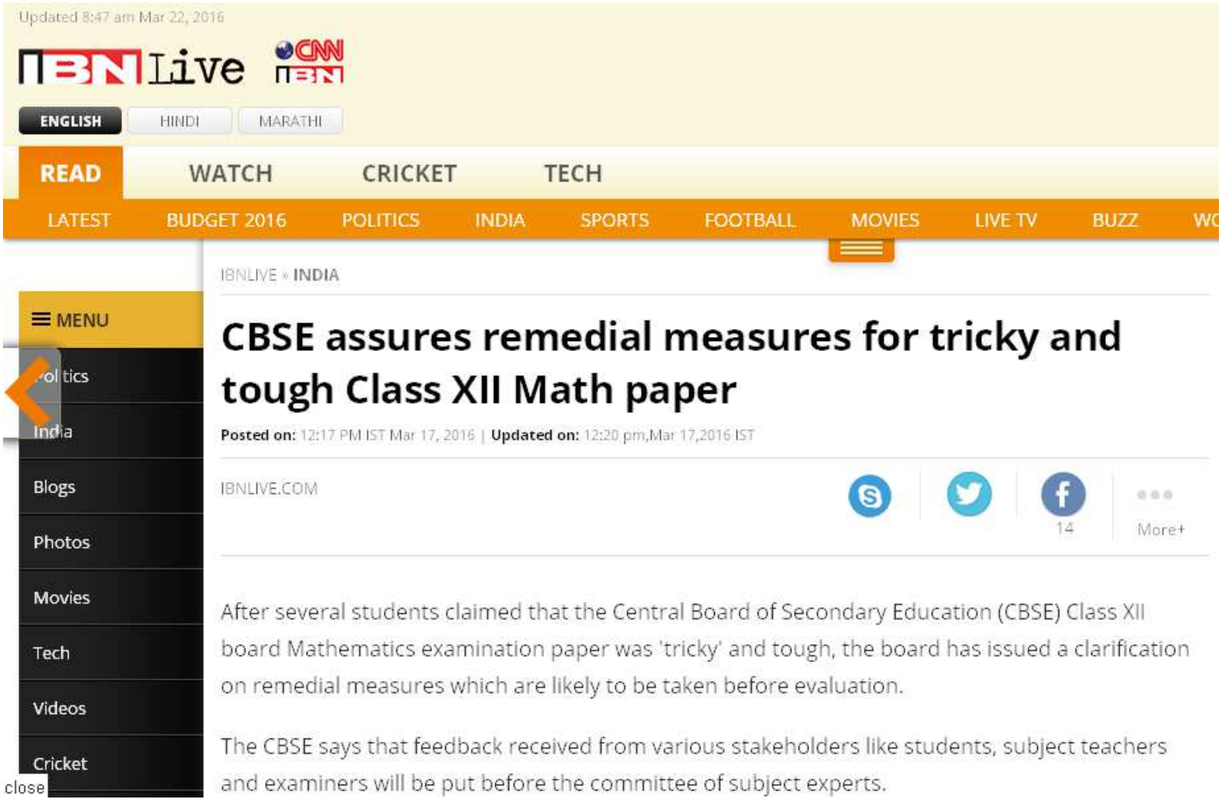
We know there can be no shoe that's fits in all.

Printed books are not e-Books! Can't be downloaded and kept in hard-disc for reading "later" ………

So if you read this book later, you will get all kinds of examples in a single place. This becomes a very good "Reference Material". I sincerely wish that all find this "very useful".

Students who do not practice lots of problems, do not do well. The rules of "doing well" had never changed …. Will never change !

After 2016 CBSE Mathematics exam lots of students complained that the paper was tough!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CBSE assures remedial measures for tricky and tough Class XII Math paper

Posted on: 12:17 PM IST Mar 17, 2016 | **Updated on:** 12:20 pm,Mar 17,2016 IST

IBNLIVE.COM

After several students claimed that the Central Board of Secondary Education (CBSE) Class XII board Mathematics examination paper was 'tricky' and tough, the board has issued a clarification on remedial measures which are likely to be taken before evaluation.

The CBSE says that feedback received from various stakeholders like students, subject teachers and examiners will be put before the committee of subject experts.

In 2015 also the same complain was there by many students

CBSE Class 12 exam: Issue of tough maths paper raised in Parliament

A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

Last Updated: Thursday, March 19, 2015 - 14:41

New Delhi: A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

In March 2016, students of Karnataka PU-II also complained the same, regarding standard 12 (PU-II Mathematics Exam). Even though the Math Paper was identical to previous year, most students had not even solved the 2015 Question Paper.

Friday, March 25, 2016 - 13:28

The NEWS Minute

HOME NEWS ANDHRA KARNATAKA KERALA TAMIL NADU TELANGANA CULTURE MEDIA BLOG

Exams

Online petition for lenient evaluation of K'taka II PU math paper gets over 8000 supporters

The campaign, which was launched on Monday, has garnered over 8000 supporters

TNM Staff | Wednesday, March 16, 2016 - 09:32

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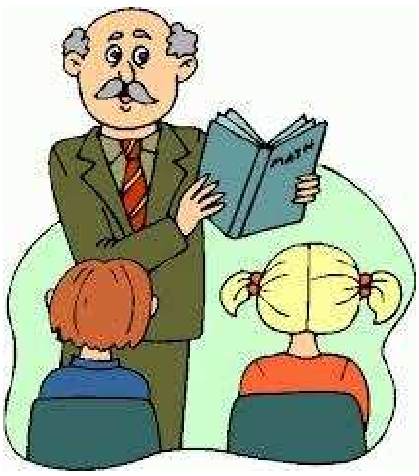
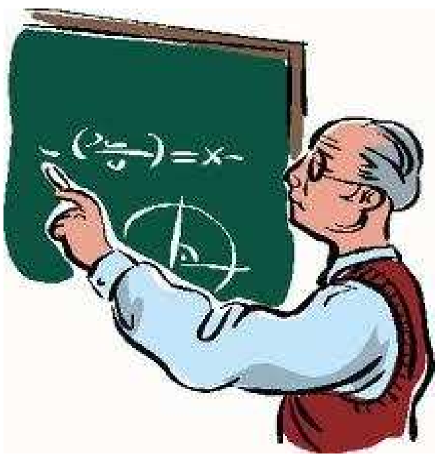
Following a "very tough" math paper that left many II PU students in tears, Saket Ravindran a student launched an online campaign demanding lenient evaluation.

These complains are not new. In fact since last 40 years, (since my childhood), I always see this; every year the same setback, same complain!

In this e-Book I am trying to solve this problem. Those students who practice can learn.

No one can help those who are not studying, or practicing.





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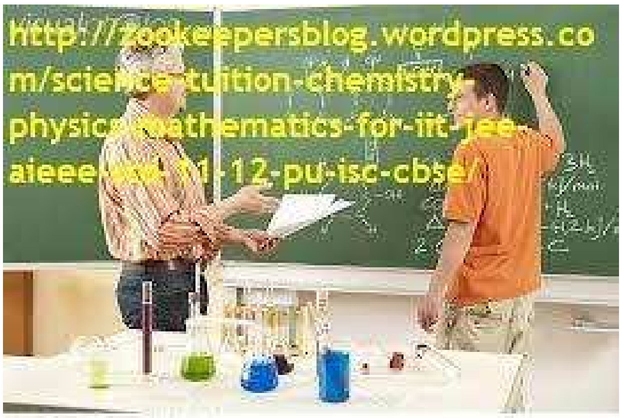


A very polite request :

I wish these e-Books are read only by Boys and Men. Girls and Women, better read something else; learn from somewhere else.

Preface

We all know that in the species “Homo Sapiens “, males are bigger than females. The reasons are explained in standard 10, or 11 (high school) Biology texts. **This shapes or size, influences all of our culture**. Before we recall / understand the reasons once again, let us see some random examples of the influence

Random - 1

If there is a Road rage, then who all fight ? (generally ?). Imagine two cars driven by adult drivers. Each car has a woman of similar age as that of the Man. The cars “ touch “ or “ some issue happens”. Who all comes out and fights ? Who all are most probable to drive the cars ?

(Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win)

Random - 2

Heavy metal music artists are all Men. Metallica, Black Sabbath, Motley Crue, Megadeth, Motorhead, AC/DC, Deep Purple, Slayer, Guns & Roses, Led Zeppelin, Aerosmith ….. the list can be in thousands. All these are grown-up Boys, known as Men.

(Men strive for perfection. Men are eager to excel. Men work hard. Men want to win.)

Random - 3

Apart from Marie Curie, only one more woman got Nobel Prize in Physics. (Maria Goeppert Mayer - 1963). So, … almost all are men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 4

The best Tabla Players are all Men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 5

History is all about, which Kings ruled. Kings, their men, and Soldiers went for wars. History is all about wars, fights, and killings by men.

Boys start fighting from school days. Girls do not fight like this

(Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win.)

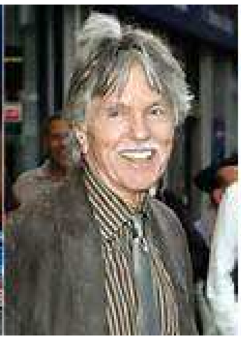
Random - 6

The highest award in Mathematics, the “ Fields Medal “ is around since decades. Till date only one woman could get that. (Maryam Mirzakhani - 2014). So, … almost all are men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 7

Actor is a gender neutral word. Could the movie like “ Top Gun “ be made with Female actors ? The best pilots, astronauts, Fighters are all Men.

Random - 8

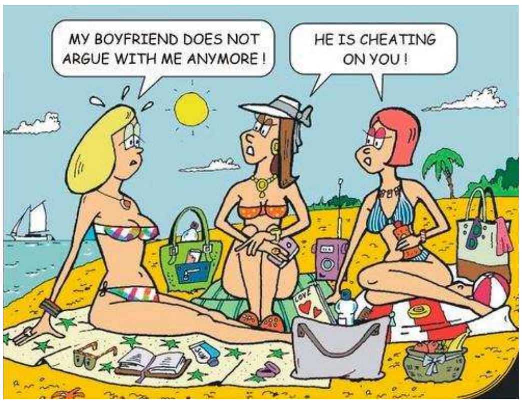
In my childhood had seen a movie named “ The Tower in Inferno “. In the movie when the tall tower is in fire, women were being saved first, as only one lift was working….


Many decades later another movie is made. A box office hit. “ The Titanic “. In this also …. As the ship is sinking women are being saved. **Men are disposable**. Men may get their turn later…

Movies are not training programs. Movies do not teach people what to do, or not to do. Movies only reflect the prevalent culture. Men are disposable, is the culture in the society. Knowingly, unknowingly, the culture is depicted in Movies, Theaters, Stories, Poems, Rituals, etc. I or you can’t write a story, or make a movie in which after a minor car accident the Male passengers keep seating in the back seat, while the both the women drivers come out of the car and start fighting very bitterly on the road. There has been no story in this world, or no movie made, where after an accident or calamity, Men are being helped for safety first, and women are told to wait.

Random - 9

Artists generally follow the prevalent culture of the Society. In paintings, sculptures, stories, poems, movies, cartoon, Caricatures, knowingly / unknowingly, “ the prevalent Reality “ is depicted. The opposite will not go well with people. If deliberately “ the opposite “ is shown then it may only become a special art, considered as a special mockery.



Random - 10

Men go to “girl / woman’s house” to marry / win, and bring her to his home. That is a sort of winning her. When a boy gets a “ Girl-Friend “, generally he and his friends consider that as an achievement. The boy who “ got / won “ a girl-friend feels proud. His male friends feel, jealous, competitive and envious. Millions of stories have been written on these themes. Lakhs of movies show this. Boys / Men go for “ bike race “, or say “ Car Race “, where the winner “ gets “ the most beautiful girl of the college.




(Men want to excel. Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win.)

Prithviraj Chauhan ‘ went ` to “ pickup “ or “ abduct “ or “ win “ or “ bring “ his love. There was a Hindi movie (hit) song … “ Pasand ho jaye, to ghar se utha laye “. It is not other way round. Girls do not go to Boy’s house or man’s house to marry. Nor the girls go in a gang to “ pick-up “ the boy / man and bring him to their home / place / den.

Random - 11

Rich people; often are very hard working. Successful business men, establish their business (empire), amass lot of wealth, with lot of difficulty. Lots of sacrifice, lots of hard work, gets into this. Rich people’s wives had no contribution in this wealth creation. Women are smart, and successful upto the extent to choose the right/rich man to marry. So generally what happens in case of Divorces ? Search the net on " most costly divorces " and you will know. The women;(who had no contribution at all, in setting up the business / empire), often gets in Billions, or several Millions in divorce settlements.

Number 1

Rupert & Anna Murdoch -- $1.7 billion

One of the richest men in the world, Rupert Murdoch developed his worldwide media empire when he inherited his father's Australian newspaper in 1952. He married Anna Murdoch in the '60s and they remained together for 32 years, springing off three children.

They split amicably in 1998 but soon Rupert forced Anna off the board of News Corp and the gloves came off. The divorce was finalized in June 1999 when Rupert agreed to let his ex-wife leave with $1.7 billion worth of his assets, $110 million of it in cash. Seventeen days later, Rupert married Wendi Deng, one of his employees.

Ted Danson & Casey Coates -- $30 million

Ted Danson's claim to fame is undoubtedly his decade-long stint as Sam Malone on NBC's celebrated sitcom Cheers . While he did other TV shows and movies, he will always be known as the bartender of that place where everybody knows your name. He met his future first bride Casey, a designer, in 1976 while doing Erhard Seminars Training.

Ten years his senior, she suffered a paralyzing stroke while giving birth to their first child in 1979. In order to nurse her back to health, Danson took a break from acting for six months. But after two children and 15 years of marriage, the infatuation fell to pieces. Danson had started seeing Whoopi Goldberg while filming the comedy, Made in America and this precipitated the 1992 divorce. Casey got $30 million for her trouble.

See https://zookeepersblog.wordpress.com/misandry-and-men-issues-a-short-summary-at-single-place/

See http://skmclasses.kinja.com/save-the-male-1761788732

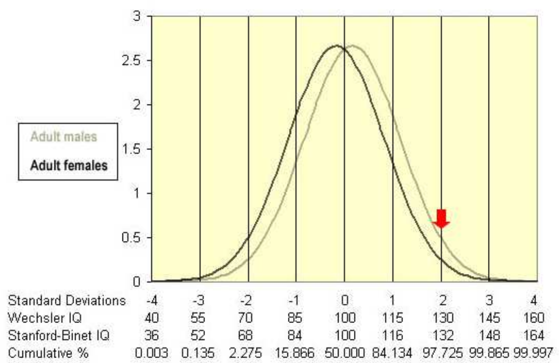
It was Boys and Men, who brought the girls / women home. The Laws are biased, completely favoring women. The men are paying for their own mistakes.

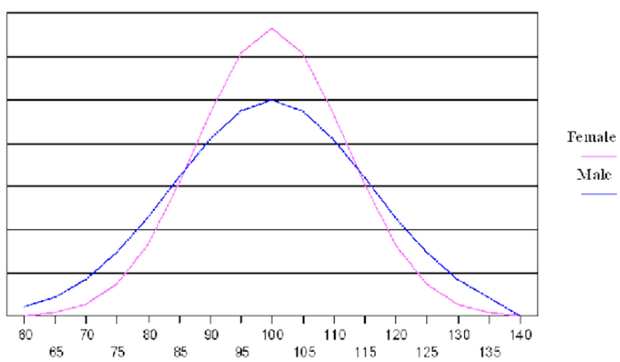
See https://zookeepersblog.wordpress.com/biased-laws/

(Man brings the Woman home. When she leaves, takes away her share of big fortune!)

Random - 12

A standardized test of Intelligence will never be possible. It never happened before, nor ever will happen in future; where the IQ test results will be acceptable by all. In the net there are thousands of charts which show that the intelligence scores of girls / women are lesser. Debates of Trillion words, does not improve performance of Girls.





I am not wasting a single second debating or discussing with anyone, on this. I am simply accepting ALL the results. IQ is only one of the variables which is required for success in life. Thousands of books have been written on " Networking Skills ", EQ (Emotional Quotient), Drive, Dedication, Focus, " Tenacity towards the end goal " … etc. In each criteria, and in all together, women (in general) do far worse than men. Bangalore is known as " ….. capital of India ". [Fill in the blanks]. The blanks are generally filled as " Software Capital ", " IT Capital ", " Startup Capital ", etc. I am member in several startup eco-systems / groups. I have attended hundreds of meetings, regarding " technology startups ", or " idea startups ". These meetings have very few women. Starting up new companies are all " Men's Game " / " Men's business ". Only in Divorce settlements women will take their goodies, due to Biased laws. There is no dedication, towards wealth creation, by women.

Random - 13

Many men, as fathers, very unfortunately treat their daughters as " Princess ". Every " non-performing " woman / wife was " princess daughter " of some loving father. Pampering the girls, in name of " equal opportunity ", or " women empowerment ", have led to nothing.

See http://skmclasses.kinja.com/progressively-daughters-become-monsters-1764484338

See http://skmclasses.kinja.com/vivacious-vixens-1764483974

There can be thousands of more such random examples, where " Bigger Shape / size " of males have influenced our culture, our Society. **Let us recall the reasons**, that we already learned in standard 10 - 11, Biology text Books. In humans, women have a long gestation period, and also spends many years (almost a decade) to grow, nourish, and stabilize the child. (Million years of habit) Due to survival instinct Males want to inseminate. Boys and Men fight for the " facility (of womb + care) " the girl / woman may provide. Bigger size for males, has a winning advantage. Whoever wins, gets the " woman / facility ". The male who is of " Bigger Size ", has an advantage to win…. Leading to Natural selection over millions of years. In general " Bigger Males "; the " fighting instinct " in men; have led to wars, and solving tough problems (Mathematics, Physics, Technology, startups of new businesses, Wealth creation, Unreasonable attempts to make things [such as planes], Hard work ….)

So let us see the IIT-JEE results of girls. Statistics of several years show that there are around 17, (or less than 20) girls in top 1000 ranks, at all India level. Some people will yet not understand the performance, till it is said that … year after year we have around 980 boys in top 1000 ranks. Generally we see only 4 to 5 girls in top 500. In last 50 years not once any girl topped in IIT-JEE advanced. Forget about Single digit ranks, double digit ranks by girls have been extremely rare. It is all about " good boys ", " hard working ", " focused ", "**Bel-esprit** " **boys**.

In 2015, Only 2.6% of total candidates who qualified are girls (upto around 12,000 rank). while 20% of the Boys, amongst all candidates qualified. The Total number of students who appeared for the exam were around 1.4 million for IIT-JEE main. Subsequently 1.2 lakh (around 120 thousands) appeared for IIT-JEE advanced.

IIT-JEE results and analysis, of many years is given at https://zookeepersblog.wordpress.com/iit-jee-iseet-main-and-advanced-results/

In Bangalore it is rare to see a girl with rank better than 1000 in IIT-JEE advanced. We hardly see 6-7 boys with rank better than 1000. Hardly 2-3 boys get a rank better than 500.

See http://skmclasses.weebly.com/everybody-knows-so-you-should-also-know.html

Professor Subhashish Chattopadhyay

Spoon Feeding Series - Differential Equation

In any book solution techniques of various types of Differential equations will be given. But in exam when you get one, you are not sure of what type is it. So you have to try the various methods one by one ……

The approach to solve Differential Equations would be as follows.

Step –1 Check if the problem is of type variable separable

If yes then solve it

Else

Step -2 Check if it is of the type exact. This is because it is easiest or fastest to solve differential equations of exact type

Else step –3 Check if the problem is modifiable to " Exact type ". (by multiplying with a I.F (Integrating Factor))

If you could identify the multiplying factor and modified then solve it as EXACT type

Else step –4 Check if some differential coefficients can be squeezed ?

Else step –5 check if it is homogeneous type ? (or is it reducible to homogeneous) ?

Else step –6 check if it linear or modifiable to linear.

Else step –7 check if it is of the form Bernoulli (This is also modifiable to linear)

Else step –8 check if it can be written as D parameter and factorized.

But before we proceed with examples and types of Differential Equations, it is important to recall the Integration rules or methods. (This chapter assumes that the students is very good at Indefinite Integral.)

-

IIT-JEE IGCSE Home Tuitions South Bangalore by Prof. Subhashish Chattopadhyay and Team of Teachers

http://iitjeeigcse.simplesite.com/

-

Recall the various tricks, formulae, and rules of solving Indefinite Integrals

(i) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(ii) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$

(iii) $\int \frac{dx}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C$

(iv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C = \cosh^{-1}\left(\frac{x}{a}\right) + C$

(vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$

(vii) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log|x+\sqrt{x^2+a^2}|\right] + C$

(viii) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + C$

(ix) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2} - a^2\log|x+\sqrt{x^2-a^2}|\right] + C$

(x) $\int (px+q)\sqrt{ax^2+bx+c}\,dx = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \left(\frac{q-pb}{2a}\right)\int \sqrt{ax^2+bx+c}\,dx$

- $\int e^x dx = e^x$
- $\int e^{ax} dx = \frac{1}{a} e^{ax}$
- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$
- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$
- $\int a^x dx = \frac{a^x}{lna} + c$

- $\int \csc x \cot x dx = -\csc x + c$
- $\int \csc^2 x dx = -\cot x + c$
- $\int \sec x \tan x dx = \sec x + c$
- $\int \sec^2 x dx = \tan x + c$
- $\int \sin x dx$ = - \cos x + c$
- $\int \cos x dx = \sin x + c$

- $\int \log x dx = x(\log x - 1) + c$
- $\int \frac{1}{x} dx = \log |x| + c$
- $\int a^x dx = a^x \log x + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$

- $\int (ax+b)^n = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$, $n \neq 1
- $\int \frac{dx}{(ax+b)} = \frac{1}{a} \log |ax+b| + C$
- $\int e^{ax+b} = \frac{1}{a} e^{ax+b} + C$
- $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$
- $\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + C$
- $\int \csc^2(ax+b) dx = \frac{-1}{a} \cot(ax+b) + C$
- $\int \csc(ax+b) \cot(ax+b) dx = \frac{-1}{a} \csc(ax+b) + C$

Some advanced procedures….

$\int \frac{x^m}{(a+bx)^p}\,dx$ m is a + ve integer	Put a + bx = z
$\int \frac{dx}{x^m (a+bx)^p}$, where either ($m$ and p positive integers) or (m and p are fractions, but $m + p$ = integers > 1)	Put a + bx = zx
$\int x^m (a+bx^n)^p\,dx$, where m, n, p are rationals.	
(i) p is a + ve integer	Apply Binomial theorem to $(a + bx^n)^p$
(ii) p is a − ve integer	Put $x = z^k$ where k = common denominator of m and n.
(iii) $\frac{m+1}{n}$ is an integer	Put $(a + bx^n) = z^k$ where k = denominator of p.
(iv) $\frac{m+1}{n} + p$ is an integer	Put $a + bx^n = x^n z^k$ where k = denominator of fraction p.

Solve a Simple Problem

$$\int \frac{3x+1}{2x^2+x+1}\,dx = \int \left(\frac{\frac{3}{4}(4x+1)+\frac{1}{4}}{2x^2+x+1} \right) dx$$

$$= \frac{3}{4}\int\left(\frac{4x+1}{2x^2+x+1}\right) dx + \frac{1}{8}\int \frac{dx}{\left(x^2+\frac{x}{2}+\frac{1}{2}\right)}$$

$$= \frac{3}{4}\log(2x^2+x+1) + \frac{1}{2\sqrt{7}}\tan^{-1}\frac{4x+1}{\sqrt{7}} + C$$

-

Solve a problem

$$\int \frac{x}{(1-x)^{1/3}-(1-x)^{1/2}}\,dx \qquad \{ \text{ The LCM of 2 and 3 is 6 } \}$$

Hence, substitute $1-x=u^6$ Then, $dx=-6u^5du$

$$\Rightarrow I = \int \frac{1-u^6}{u^2-u^3}(-6u^5du) = -6\int \frac{1-u^6}{1-u}u^3du$$

$$= -6\int (1+u+u^2+u^3+u^4+u^5)\,u^3du$$

$$= -6\left(\frac{1}{4}u^4+\frac{1}{5}u^5+\frac{1}{6}u^6+\frac{1}{7}u^7+\frac{1}{8}u^8+\frac{1}{9}u^9\right)+c$$

Solve a Problem

Evaluate $\int \cos 2x \log(1+\tan x)dx$.

Solution:

Integrating by parts taking cos 2x as the 2nd function, the given integral

$$= \{\log(1+\tan x)\}\frac{\sin 2x}{2} - \int \frac{\sec^2 x}{1+\tan x}\cdot\frac{\sin 2x}{2}dx$$

$$= \frac{1}{2}\sin 2x \log(1+\tan x) - \int \frac{\sin x}{\sin x + \cos x}dx.$$

Now $\int \frac{\sin x\, dx}{\sin x + \cos x}$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x}dx,$$

$$= \frac{1}{2}\int \left[1 - \frac{\cos x - \sin x}{\sin x + \cos x}\right]dx = \frac{1}{2}[x - \log(\sin x + \cos x)].$$

Hence the given integral

$$= \frac{1}{2}\sin 2x \log(1+\tan x) - \frac{1}{2}[x - \log(\sin x + \cos x)].$$

Recall how to integrate Linear X root Quadratic in denominator

Find the value of the $\int \frac{dx}{(x+1)\sqrt{(1+2x-x^2)}}$

Putting $(x+1) = \frac{1}{t}$, so that $dx = -\frac{1}{t^2}\,dt$, $x = \frac{1-t}{t}$ and

$$(1+2x-x^2) = 1 + 2\left(\frac{1-t}{t}\right) - \frac{(1-t)^2}{t^2} = \frac{2}{t^2}\left[\left(\frac{1}{\sqrt{2}}\right)^2 - (t-1)^2\right],$$

we get the value of the given integral transformed as'

$$\int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\frac{2}{\sqrt{t}}\left[\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2\right]} = -\frac{1}{\sqrt{2}}\sin^{-1}\frac{t-1}{\left(\frac{1}{\sqrt{2}}\right)} + C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}\,x}{(x+1)} + C$$

Another advanced example

Example Evaluate $\int \frac{dx}{x\sqrt{(1+x^n)}}$

Make the substitution $(1 + x^n) = t^2$ or $x^n = (t^2 - 1)$, so that $n\, x^{n-1}\, dx = 2t\, dt$, we get

$$\int \frac{2t\,dt}{n x^n t} = \frac{2}{n}\int \frac{dt}{(t^2-1)} = \frac{1}{n}\ln\left|\frac{t-1}{t+1}\right|$$

$$= \frac{1}{n}\ln\left|\frac{\sqrt{(1+x^n)}-1}{\sqrt{(1+x^n)}+1}\right| + C$$

Similarly

The value of integral $\int \frac{dx}{x\sqrt{1-x^3}}$ is given by

(a) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}+1}{\sqrt{1-x^3}-1}\right|+C$ (b) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^2}+1}\right|+C$

(c) $\frac{2}{3}\log\left|\frac{1}{\sqrt{1-x^3}}\right|+C$ (d) $\frac{1}{3}\log|1-x^3|+C$

Ans. (b)

Solution Put $1-x^3=t^2$. Then $-3x^2dx=2t\,dt$ and the integral becomes

$$-\frac{1}{3}\int\frac{-3x^2dx}{x^3\sqrt{1-x^3}}=-\frac{1}{3}\int\frac{2t\,dt}{(1-t^2)t}=\frac{2}{3}\int\frac{dt}{t^2-1}$$

$$=\frac{2}{3}\left(\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|\right)+C=\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+C$$

Solve a Problem

$\int\sqrt{\sec x-1}\;dx$ is equal to

(a) $2\log\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(b) $\log\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(c) $-2\log\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(d) none of these

(c). $\int \sqrt{\sec x - 1}\ dx = \int \sqrt{\frac{1-\cos x}{\cos x}}\ dx$

$$= \sqrt{2}\int \frac{\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{x}{2} - 1}}\ dx = -2\sqrt{2}\int \frac{dz}{\sqrt{2z^2-1}}$$

$$\left(\text{Putting } \cos\frac{x}{2} = z \Rightarrow \sin\frac{x}{2}\ dx = -2dz\right)$$

$$= -2\int \frac{dz}{\sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -2\log\left[z + \sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2}\right] + C$$

$$= -2\log\left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$$

Solve another problem

$$I = \int \sqrt{1+\operatorname{cosec} x}\,.\,dx$$

$$= \int \sqrt{1+\frac{1}{\sin x}}\,.\,dx = \int \sqrt{\frac{\sin x + 1}{\sin x}}\,.\,dx$$

$$= \int \sqrt{\frac{(1+\sin x)(1-\sin x)}{\sin x\,(1-\sin x)}}\,.\,dx \qquad \text{[On rationalization]}$$

$$= \int \sqrt{\frac{1-\sin^2 x}{\sin x - \sin^2 x}}\,.\,dx \qquad [\because \quad (a+b)(a-b) = a^2 - b^2]$$

$$= \int \frac{\cos x}{\sqrt{\sin x - \sin^2 x}}\,.\,dx \qquad [\because \quad \sin^2 A + \cos^2 A = 1]$$

$$\sin x = z \quad \Rightarrow \quad \cos x\,dx = dz$$

$$I = \int \frac{1}{\sqrt{z - z^2}}\,.\,dz = \int \frac{1}{\sqrt{-(z^2 - z)}}\,.\,dz$$

$$= \int \frac{1}{\sqrt{\frac{1}{4} - \left(z^2 - z + \frac{1}{4}\right)}}\,.\,dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(z - \frac{1}{2}\right)^2}}\,.\,dz$$

$$\left(z - \frac{1}{2}\right) = y \quad \Rightarrow \quad dz = dy$$

$$I = \int \frac{1}{\sqrt{(1/2)^2 - y^2}}\,.\,dy \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}}\,.\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \sin^{-1}\left(\frac{y}{1/2}\right) + c$$

$$= \sin^{-1}\left(\frac{z - 1/2}{1/2}\right) + c \qquad [\because \quad y = z - 1/2]$$

Solve another Integral

$$I = \int \sqrt{\frac{1+x}{x}} \, . \, dx$$

$$= \int \sqrt{\frac{1+x}{x} \times \frac{1+x}{1+x}} \, dx \qquad \text{[Multiply and divided by } (1+x)]$$

$$= \int \sqrt{\frac{(1+x)^2}{x(1+x)}} \, . \, dx = \int \frac{1+x}{\sqrt{x+x^2}} \, . \, dx$$

Let us write :

$$1 + x = \lambda \, . \, \frac{d}{dx}(x + x^2) + \mu$$

$$\Rightarrow \quad 1 + x = \lambda(1 + 2x) + \mu \qquad \text{...(1)}$$

$$\Rightarrow \quad 1 + x = 2\lambda x + \lambda + \mu$$

Comparing the coefficients of x and the constant terms, we have

$$1 = 2\lambda \;\Rightarrow\; \lambda = \frac{1}{2}$$

and

$$1 = \lambda + \mu \;\Rightarrow\; \mu = 1 - \lambda = 1 - \frac{1}{2} = \frac{1}{2}\,.$$

Putting the values of λ and μ in (1),

$$1 + x = \frac{1}{2}(1+2x) + \frac{1}{2}\,.$$

$$\therefore \quad I = \int \frac{\frac{1}{2}(1+2x) + \frac{1}{2}}{\sqrt{x+x^2}} \, . \, dx$$

$$= \frac{1}{2}\int \frac{1+2x}{\sqrt{x+x^2}} \, dx + \frac{1}{2}\int \frac{1}{\sqrt{x+x^2}} \, . \, dx$$

$$\Rightarrow \quad I = \frac{1}{2} \, I_1 + \frac{1}{2} \, I_2 \qquad \text{...(2)}$$

Now

$$I_1 = \int \frac{1+2x}{\sqrt{x+x^2}} \, dx$$

Put

$$x + x^2 = z \;\Rightarrow\; (1 + 2x)\, dx = dz$$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \, . \, dz = \int z^{-1/2} \, . \, dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2\sqrt{z} + c_1$$

$$= 2\sqrt{x + x^2} + c_1 \qquad \text{...(3)}$$

and

$$I_2 = \int \frac{1}{\sqrt{x+x^2}} \, . \, dx$$

$$= \int \frac{1}{\sqrt{\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4}}} \cdot dx \qquad \left[\begin{array}{l} \text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4} \end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \cdot dx$$

Put $\quad x + \frac{1}{2} = z \Rightarrow dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{z^2 - \left(\frac{1}{2}\right)^2}} \cdot dz \qquad \left[\text{By using } \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c\right]$$

$$= \log\left|z + \sqrt{z^2 - \left(\frac{1}{2}\right)^2}\right| + c_2 = \log\left|\left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}\right| + c_2$$

$$= \log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right| + c_2 \qquad \text{...(4)}$$

$\therefore$ **From equation (2),**

$$I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \text{[Using (3) and (4)]}$$

Solve another problem

$$I = \int \frac{ax^3 + bx}{x^4 + c^2}\, dx = \int \frac{ax^3}{x^4 + c^2} . dx + \int \frac{bx}{x^4 + c^2} . dx$$

$$= a \int \frac{x^3}{x^4 + c^2} . dx + b \int \frac{x}{x^4 + c^2} . dx$$

$$\Rightarrow \quad I = a\, I_1 + b\, I_2 \qquad ...(1)$$

Now $$I_1 = \int \frac{x^3}{x^4 + c^2} . dx$$

$$= \frac{1}{4} \int \frac{4x^3}{x^4 + c^2} . dx$$ [Multiply and divided by 4]

$$= \frac{1}{4} \log \left| x^4 + c^2 \right| + c_1 \qquad ...(2) \quad \left[\because \int \frac{f'(x)}{f(x)} . dx = \log |f(x)| + c \right]$$

and $$I_2 = \int \frac{x}{x^4 + c^2} . dx$$

$$= \frac{1}{2} \int \frac{2x}{(x^2)^2 + c^2}\, dx$$ [Multiply and divided by 2]

Put $x^2 = z \;\Rightarrow\; 2x\, dx = dz$

$$= \frac{1}{2} \int \frac{1}{z^2 + c^2}\, dz \qquad \left[\text{By using} \int \frac{1}{x^2 + a^2} . dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \right]$$

Solve Integration root linear plus root linear in denominator

If $I = \int \frac{dx}{\sqrt{2x+3}+\sqrt{x+2}}$, then I equals

(a) $2(u - v) + \log\left|\frac{u-1}{u+1}\right| + \log\left|\frac{v-1}{v+1}\right| + C$

$u = \sqrt{2x+3},\ v = \sqrt{x+2}$

(b) $\log\left|\frac{\sqrt{x+2}+\sqrt{2x+3}}{\sqrt{x+2}-\sqrt{2x+3}}\right| + C$

(c) $\log\left(\sqrt{x+2}+\sqrt{2x+3}\right) + C$

(d) is transcedental function in u and v, $u = \sqrt{2x+3}$

$v = \sqrt{x+2}$

Ans. (a), (d)

$$I = \int \frac{\sqrt{2x+3} - \sqrt{x+2}}{x+1}\,dx$$

$$= I_1 - I_2$$

where $I_1 = \int \frac{\sqrt{2x+3}}{x+1}\,dx$ and $I_2 = \int \frac{\sqrt{x+2}}{x+1}\,dx$

Put $2x + 3 = t^2$, in I_1, so that

$$I_1 = \int \frac{2t \cdot t}{t^2 - 1}\,dt = 2\int\left[1 + \frac{1}{t^2-1}\right]dt$$

$$= 2\left[t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|\right]$$

In I_2, put $x + 2 = y^2$, so that

$$I_2 = \int \frac{2y^2}{y^2-1}\,dy = 2y + \log\left|\frac{y-1}{y+1}\right|$$

Thus,

$$I = 2\left(\sqrt{2x+3} - \sqrt{x+2}\right) + \log\left|\frac{\sqrt{2x+3}-1}{\sqrt{2x+3}+1}\right| + \log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$$

Solve another Problem

Evaluate $\int \frac{\sin 2x\, dx}{(a + b\cos x)^2}$.

Solution:

We have $I = \int \frac{\sin 2x dx}{(a+b\cos x)^2} = 2\int \frac{\sin x \cos x dx}{(a+b\cos x)^2}$

Now put $a + b \cos x = t$

so that $-b \sin x\, dx = dt$.

Also $\cos x = \frac{(t-a)}{b}$.

$$\therefore\ I = -\frac{2}{b}\int \frac{(t-a)/b}{t^2} dt = -\frac{2}{b^2}\int\left[\frac{t}{t^2} - \frac{a}{t^2}\right]dt$$

$$= -\frac{2}{b^2}\int\left[\frac{1}{t} - \frac{a}{t^2}\right]dt = -\frac{2}{b^2}\left[\log t + \frac{a}{t}\right]$$

$$= -\frac{2}{b^2}\left[\log(a+b\cos x) + \frac{a}{a+b\cos x}\right].$$

A special Integral

$$\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{(1+x+x^2)}}\,dx$$

Here we set $\sqrt{1+x+x^2} = xt + 1$, so that

$$x = \frac{2t-1}{1-t^2},\ dx = \frac{2t^2-2t+2}{(1-t^2)^2}\,dt \text{ and}$$

$$(1-\sqrt{1+x+x^2}) = \frac{-2t^2+t}{(1-t^2)}$$

Substitution of these values in the given integral transforms the problem in the form

$$\int \frac{(-2t^2+t)^2\,(1-t^2)^2\,(1-t^2)\,(2t^2-2t+2)}{(1-t^2)^2\,(2t-1)^2\,(t^2-t+1)\,(1-t^2)^2}\,dt$$

$$= +2\int \frac{t^2}{1-t^2}\,dt = -2t + \ln\left|\frac{1+t}{1-t}\right| + C$$

An advanced example

$$I=\int\frac{(x+1)}{x(1+xe^x)^2}\,dx$$

$$I=\int\frac{e^x(x+1)}{x\,e^x(1+xe^x)^2}\,dx$$

put $1 + xc^x = t,\ (xe^x + e^x)\ dx = dt$

$$I=\int\frac{dt}{(t-1)t^2}=\int\left(\frac{1}{1-t}+\frac{1}{t}+\frac{1}{t^2}\right)dt$$

$$=-\log|1-t|+\log|t|-\frac{1}{t}+C=\log\left|\frac{t}{1-t}\right|-\frac{1}{t}+C$$

$$=\log\left|\frac{1+xe^x}{-xe^x}\right|-\frac{1}{1+xe^x}+C=\log\left(\frac{1+xe^x}{xe^x}\right)-\frac{1}{1+xe^x}+C$$

Order and Power of a Differential Equation

Consider the given differential equation, $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\left(c\frac{d^2y}{dx^2}\right)^{\frac{1}{3}}$

Squaring on both the sides, we have

$$1+\left(\frac{dy}{dx}\right)^2=\left(c\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}$$

Cubing on both the sides, we have

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left\{\left(c\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}\right\}^3$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^6+3\left(\frac{dy}{dx}\right)^2+3\left(\frac{dy}{dx}\right)^4=c^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow c^2\left(\frac{d^2y}{dx^2}\right)^2-\left(\frac{dy}{dx}\right)^6-3\left(\frac{dy}{dx}\right)^4-3\left(\frac{dy}{dx}\right)^2-1=0$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ *and its power is* 2.

Therefore, the given differential equation is a non − linear differential equation of second order and second degree.

Another Example

Consider the given differential equation,

$$\sqrt[3]{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx}}$$

Cubing on both the sides of the above equation, we have

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^{\frac{3}{2}}$$

Squaring on both the sides of the above equation, we have

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[\left(\frac{dy}{dx}\right)^{\frac{3}{2}}\right]^2$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left[\left(\frac{dy}{dx}\right)\right]^3$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \left[\left(\frac{dy}{dx}\right)\right]^3 = 0$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$

and its power is 2.

Therefore, the given differential equation is a non – linear differential equation of second order and second degree.

We will see more examples at the end of the Chapter

Form the Differential equation by eliminating the unknown constants

$$xy = ae^{x} + be^{-x} + c$$

Sol : $xy = ae^{x} + be^{-x} + c$

$$\therefore x\frac{dy}{dx} + y = ae^{x} - be^{-x}$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^{x} + be^{-x}$$

$$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - c$$

$$\therefore x\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$$

$$\therefore x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$$ is the differential equation.

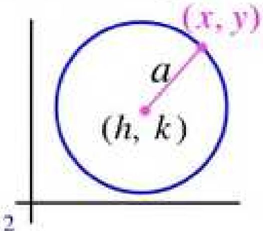
Another common example to find the Differential Equation

Find the differential equation of all circles of radious 'a' in a plane.

Sol: The family of circles of radious 'a' in a plane is

$$(x-h)^2+(y-k)^2=a^2 \longrightarrow (1)$$

where h, k are parameters.



$$\therefore \cancel{2}(x-h)+\cancel{2}(y-k)\frac{dy}{dx}=0 \longrightarrow (2)$$

$$\therefore 1+(y-k)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}=0 \Rightarrow y-k=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \longrightarrow (3)$$

$$\therefore \text{From (2)}, \ x-h=-(y-k)\frac{dy}{dx}=\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx} \longrightarrow (4)$$

Substituting (3) and (4) in (1) we get

Substituting (3) and (4) in (1) we get

$$y-k=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \longrightarrow (3)$$

$$\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx}\right]^2+\left[-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^2=a^2.$$

$$x-h=\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx} \longrightarrow (4)$$

$$\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=a^2.$$

$$(x-h)^2+(y-k)^2=a^2 \longrightarrow (1)$$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^3}{\left(\frac{d^2y}{dx^2}\right)^2}=a^2$$ is the differential equation.

Now let us see Differential Equation types

1) Let us see an example of variable separable type

Solve $\dfrac{dy}{dx} = 1 + x + y + xy$

Sol: Given $\dfrac{dy}{dx} = 1 + x + y + xy$

$$= 1 + x + y(1 + x)$$

$$= (1 + x)(1 + y)$$

$$\therefore \frac{dy}{1 + y} = (1 + x)dx$$

$$\therefore \int \frac{dy}{1 + y} = \int (1 + x)dx + c$$

$$\therefore \log(1 + y) = x + \frac{x^2}{2} + c$$

Which is the required solution.

1.1 > Solve $v\dfrac{dv}{dx} + \dfrac{\mu}{x^2} = 0$

So $\int$ v dv + $\mu \int \dfrac{dx}{x^2}$ = C

1.2 > Solve

$$\left(1 + x^2\right) dy = \sqrt{y} \cdot dx$$

So $\dfrac{dx}{1 + x^2}$ = $\dfrac{dy}{\sqrt{y}}$ Integrate both sides. You get

$$2\sqrt{y} - \tan^{-1} x = C$$

1.3 > Solve $y\,dx + x\,dy = 0$

Divide throughout by xy and we get

$\frac{dx}{x} + \frac{dy}{y} = 0$ so $\int \frac{dx}{x} + \int \frac{dy}{y} = C$

(So Multiplying factor or Integrating factor is 1 / xy)

So solution are

$\log xy = \log e^C$, i.e., $xy = e^C$; or $\log xy = \log C'$, i.e., $xy = C'$

1.4 > Solve

$(x - y^2)dx + 2xy\,dy = 0$

Will become variable separable after substitution

Substitute v = y^2 and then divide by x^2

will give $dx/x + d(v/x) = 0$

1.5 > Modifiable to variable separable by substitution

dy / dx = root (y-x)

Now as is it is not variable separable

Put y-x = u^2 so dy/dx - 1 = 2u (du / dx)

So 2u (du / dx) = u - 1

=> dx = (2u / (u -1)) du = 2 (1 + (1 / (u - 1)) du

=> x + c = 2 (u + ln |u-1|)

=> x + c = 2 (root (y-x) + ln |root(y-x) - 1 |

Reducible to Variable Separable

$$\frac{dy}{dx} = f(ax+by+c)$$

Let $ax+by+c = v$

Solve $\frac{dy}{dx} = [3x+y+4]^2$

Sol: Given $\frac{dy}{dx} = [3x+y+4]^2 \longrightarrow (1)$

Let $3x+y+4 = v \longrightarrow (2)$

$$\therefore 3 + \frac{dy}{dx} = \frac{dv}{dx} \longrightarrow (3)$$

Eliminate y, by substituting (2),(3) in (1)

$$\frac{dv}{dx} - 3 = v^2$$

$$\therefore \frac{dv}{dx} = v^2 + 3$$

$$\therefore \frac{dv}{v^2+3} = dx$$

$$\therefore \int \frac{dv}{v^2+3} = \int dx + c$$

$$\therefore \frac{1}{\sqrt{3}} \tan^{-1} \frac{v}{\sqrt{3}} = x + c$$

The solution of the given d.e. is

$$\frac{1}{\sqrt{3}} \tan^{-1} \frac{3x+y+4}{\sqrt{3}} = x + c$$

2) Exact type

A differential equation written in the form

$M\,dx + N\,dy = 0$

is Exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

2.1 > Solve $(x+2y)\,x\,dx + (x^2 - y^2)\,dy = 0$

Observe $M = x\,(x+2y)$ and $N = (x^2 - y^2)$

Also it is already in Mdx + N dy = 0 form (Else before testing M and N it has to brought to left)

$\partial M/\partial dy = 2x$ and $\partial N/\partial x = 2x$

So it Exact.

Solution is $\int$ M ∂x + $\int$ (Those terms of N without x) dy = C

So $x^3 / 3 + 2y\,x^2 / 2 + (-y^3) / 3 = c$

In M 2y is treated as constant and $-y^2$ is only taken out of N

Finally $\frac{1}{3}x^3 + x^2 y - \frac{1}{3}y^3 = C$

2.2 > Solve $(xy^2 + x)\,dx + yx^2\,dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[xy^2 + x] = 2xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[yx^2] = 2xy$$

So Solution is $\int (xy^2 + x) \;\; \partial x$ + $\int$ (0) dy = c (Note there are no terms in N without x)

=> y^2 (x^2 /2) + (x^2 / 2) = c

3) Ways to convert an equation, which is not Exact type to Exact by Guessing the Multiplying factor

$y\,dx - x\,dy = 0$ is not Exact type

We can guess that multiplying by x^{-2}, $x^{-1}y^{-1}$, or y^{-2} Changes this to Exact. (So it is just an intelligent guess)

Other Guesses

Since $d(x^m y^n) = x^{m-1}y^{n-1}(my\,dx + nx\,dy)$

It has I.F of the form $x^{m-1}y^{n-1}$

$x^{km-1-\alpha}y^{kn-1-\beta}$ is an integrating Factor for any value of K

3.1 > Solve

$y^3(y\,dx - 2x\,dy) + x^4(2y\,dx + x\,dy) = 0$

As given it is not exact. But by multiplying throughout by x^{-3} It becomes Exact

Check if the final answer is $2x^4y - y^4 = Cx^2$

3.2 > $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy$ = 0 seems to have xy as Integrating Factor

If the equation is of the form $f_1(x, y)\,y\,dx + f_2(x, y)\,x\,dy = 0$

Then 1 / $(Mx - Ny)$ is an Integrating factor

$(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$

has 1 / 2xy as the I.F

It is difficult to remember the following

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a pure function of x then $e^{\int f(x)dx}$

Or $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a pure function of y then $e^{\int f(y)dy}$

4) Can we squeeze the differential coeffs ?

(Special Exact Differentials)

d(xy) = (dx)y + x (dy)

d(ln |xy|) = ((dx)y+ x(dy))/xy

$d(x^2+y^2) = 2\,x\,dx + 2\,y\,dy$

or $x\,dx + y\,dy = (½)d(x^2+y^2)$

$d(x^2y^2) = 2x\ dx\ y^2 + x^2\ 2\ y\ dy = 2xy\ (y\ dx + x\ dy)$

So $y\ dx + x\ dy = (1 / 2xy)\ d (x^2y^2)$

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$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{y^2} = -d\left(\frac{x}{y}\right)$$

$$\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\frac{xdx + ydy}{x^2 + y^2} = d\left(\frac{1}{2}\ln\left(x^2 + y^2\right)\right)$$

$$\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = d\left(\sqrt{x^2 + y^2}\right)$$

$$\frac{xdx - ydy}{\sqrt{x^2 - y^2}} = d\left(\sqrt{x^2 - y^2}\right)$$

Don't forget

☺ dx + dy = d(x + y)

dx - dy = d(x - y)

4.1 > Solve

$$xdx + (y - \sqrt{x^2 + y^2})dy = 0$$

Reorganize as $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} = dy$

$$d\left(\sqrt{x^2 + y^2}\right) = dy$$

So $\sqrt{x^2 + y^2} = y + c$

4.2 > Solve

$$\left(x^2 + y^2 + y\right)dx - xdy = 0$$

Reorganize to write as $dx + \dfrac{ydx - xdy}{x^2 + y^2} = 0$

=> dx = $d\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right)$

So $x - \tan^{-1}\left(\frac{y}{x}\right) = c$

5) If powers of all terms are same in Numerator and Denominator then homogeneous. Put y= vx Differentiate and proceed

5.1 > Solve $\frac{dy}{dx}$ = (2x+3y) / (4x + 5y)

Put y = vx

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ = (2x + 3vx) / (4x + 5vx) see x cancels out

=> $v + x\frac{dv}{dx}$ = (2 + 3v) / (4 + 5v) This variable separable type and gets solved easily

5.2 > Solve $\frac{dy}{dx} = \frac{x+y}{x}$

Put y = vx

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

So $v + x\frac{dv}{dx} = \frac{x+vx}{x} = 1 + v$

=> $x\frac{dv}{dx} = 1$

=> $\int dv = \int \frac{dx}{x}$

=> v = Ln x + c or ln x + ln c = ln (xc)

5.3 >

Solve $\frac{dy}{dx} = \frac{xy + x^2}{y^2}$

Put y = vx

=> $\frac{dy}{dx} = v + x\frac{dv}{dx}$ = (xvx + x^2) / (v^2 x^2) (x^2 cancels out)

$v + x\frac{dv}{dx}$ = (v + 1) / v^2

This is variable separable type

Let us discuss an example of modifiable to Homogeneous form

Solve $2x+y+1\ dx+\ 4x+2y-1\ dy=0$

Sol: Given $2x+y+1\ dx+\ 4x+2y-1\ dy=0$

$$\therefore \frac{dy}{dx}=-\frac{2x+y+1}{4x+2y-1}.\ \text{Here } \frac{2}{4}=\frac{1}{2}$$

$$\therefore \frac{dy}{dx}=-\frac{2x+y+1}{2\ 2x+y\ -1} \longrightarrow (1)$$

$$\text{Let } 2x+y=v \longrightarrow (2)$$

$$\text{sothat } 2+\frac{dy}{dx}=\frac{dv}{dx} \longrightarrow (3)$$

Eliminate y, by substituting (2),(3) in (1)

$$\frac{dv}{dx}-2=-\frac{v+1}{2v-1}$$

$$\therefore \frac{dv}{dx}=2-\frac{v+1}{2v-1}$$

$$\therefore \frac{dv}{dx}=2-\frac{v+1}{2v-1}=\frac{4v-2-v-1}{2v-1}=\frac{3v-3}{2v-1}$$

$$\therefore \frac{2v-1}{v-1}dv=3dx$$

$$\therefore 2v+\log|v-1|=3x+c$$

$\therefore$ The required solution is

$$2[2x+y]+\log|2x+y-1|=3x+c$$

$$i.e.,\ x+2y+\log|2x+y-1|=c$$

Next example is more advanced example of modifiable to Homogeneous

5.4 >
Modifiable to Homogeneous
Solve ($2x^2 + 3y^2 - 7$)x dx - ($3x^2 + 2y^2 - 8$) y dy = 0
As is this is not homogeneous
But put $x^2 = u$ and $y^2 = v$

So (2u +3v - 7) du - (3u + 2v - 8) dv = 0
By technique described below this can be Reduced or modified to homogeneous

Reducible to Homogeneous

$$(ax + by + c)\,dx + (a'x + b'y + c')\,dy = 0.$$

Put v = x+h and y = w + k
Find h and k such that constants become zero

h = $\dfrac{b'c - bc'}{a'b - ab'}$ k = $\dfrac{ac' - a'c}{a'b - ab'}$

5.5 > Solve $(3y - 7x - 7)\,dx + (7y - 3x - 3)\,dy = 0$

Put y = Y + h and x = X + k search h and k such that constants are zero
h = -1 and k = 0

So Y = v X will solve this

6) Linear

$$\frac{dy}{dx} + p(x)y = q(x)$$

The Integrating factor is exp($\int$ p dx)

$$ye^{\int p dx} = \int Q e^{\int P dx} dx + C$$ where c is the constant of integration

Solve $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1$

Sol: Given equation can be written as

$$\frac{e^{2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$ which is linear.

Here $P = \dfrac{1}{\sqrt{x}}$, $Q = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \int P dx = \int \frac{1}{\sqrt{x}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{x}$$

$$\therefore IF = e^{\int P dx} = e^{2\sqrt{x}}$$

$\therefore$ The solution is

$$y \; IF = \int Q \; IF \; dx + c$$

$$y \cdot e^{2\sqrt{x}} = \int \frac{e^{2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}} dx + c$$

i.e., $y \cdot e^{2\sqrt{x}} = \displaystyle\int \frac{1}{\sqrt{x}} dx + c$

i.e., $y \cdot e^{2\sqrt{x}} = 2\sqrt{x} + c$

6.1 > Solve $x\frac{dy}{dx} + y = 3x^2$

Convert to P(x) , Q(x) form

$$\frac{dy}{dx} + \frac{1}{x}y = 3x$$

=> $P = \frac{1}{x}, Q = 3x$

=> I.F = $\int P\,dx$ = $\int \frac{1}{x}\,dx$ = Ln x

$$e^{\int P dx} = e^{\log x} = x$$

So solution is $y . e^{\int P dx} = \int Q . e^{\int P dx} . dx + C$

=> $y . x = \int 3x\,(x)dx + C$

=> $xy = \frac{3x^3}{3} + C$

=> $xy = x^3 + C$

6.2 >

dy/dx + (tan x) y = $\cos^2(x)$

I.F = $e^{\int \tan(x)dx} = e^{-\ln(\cos(x))} = e^{\ln(\sec(x))} = \sec(x)$

Multiply with Sec x throughout we get RHS as

$$\int \sec(x)\cos^2(x)dx = \int \cos(x)dx = \sin(x)$$

So Solution is $y = \frac{\sin(x) + C}{\sec(x)} = \left(\sin(x) + C\right)\cos(x)$

Modifiable to Linear

6.3 > $y y' \text{Sin } x = (\text{Sin } x - y^2) \text{Cos } x$

Put $y^2 = u \Rightarrow 2 y y' = du / dx$
$((\text{Sin } x) / 2) (du / dx) = (\text{Sin } x - u) \text{Cos } x$
$\Rightarrow (du / dx) + 2u \text{ Cot } x = 2 \text{ Cos } x$
This is linear

So Integrating Factor is $\exp (2 \int \text{Cot } x \, dx)$
$= \exp (2 \ln \sin x) = \text{Sin}^2 x$

$\Rightarrow \text{Sin}^2 x (du / dx) + 2 u \text{ Sin } x \text{ Cos } x = 2 \text{ Cos } x \text{ Sin}^2 x$

$\Rightarrow d/dx \text{ of } (u \text{ Sin}^2 x) = d / dx \text{ of } (2 \text{ Sin}^3 x) / 3$

$\Rightarrow u \text{ Sin}^2 x = (2/3) \text{ Sin}^3 x + c$

$\Rightarrow y^2 = (2 / 3) \text{ Sin } x + c \text{ Cosec}^2 x$

6.4 > Solve $dy / dx = (2xy) / (x^2 - 1 - 2y)$

Rewrite $2x (dx / dy) - (x^2 / y) = -(1 + 2y) / y$

Put $x^2 = u$ So $2x (dx / dy) = (du / dy)$

$\Rightarrow du / dy - (1 / y) u = - (1 + 2 y) / y$

This is linear in u and y

Integrating Factor exp(- $\int$ (1 / y) dy = exp (-ln |y|) = 1 / y

(1 / y) (du / dy) - u / y^2 = - (1 + 2y) / y^2

=> d/dy (u /y) = -1 / y^2 - 2/y

=> u / y = 1/y - 2 ln y + c

=> x^2 / y = 1 / y - 2 ln y + c

7) Bernoulli's Eqn $dy/dx + Py = Qy^n$

Divide by y^n and multiply by 1-n

We get $\frac{1-n}{y^n}\frac{dy}{dx} + (1-n)\,Py^{1-n} = (1-n)\,Q$

Put $y^{n-1} = v$ changes to dv/dx + (1-n)Pv = Q(1-n)

7.1 > Solve

$$dy/dx + x\sin 2y = x^3\cos^2 y$$

Divide by $\cos^2 y$ Put Tan y = v

Solve $x\dfrac{dy}{dx} + y = x^2 y^6$

Sol: Given equation can be written as

$\dfrac{dy}{dx} + \dfrac{1}{x}y = xy^6$, Which is Bernoulli's equation.

$$\therefore y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x \longrightarrow (1)$$

Let $y^{-5} = v \longrightarrow (2)$

sothat $-5y^{-6}\dfrac{dy}{dx} = \dfrac{dv}{dx} \longrightarrow (3)$

Eliminate y, by substituting (2),(3) in (1)

$$\frac{1}{-5}\frac{dv}{dx} + \frac{1}{x}v = x$$

$$\frac{1}{-5}\frac{dv}{dx} + \frac{1}{x}v = x$$

$\therefore \dfrac{dv}{dx} + \dfrac{-5}{x}v = -5x$ which is linear.

Here $P = \dfrac{-5}{x}$, $Q = -5x$

$$\therefore \int P\,dx = \int \frac{-5}{x}dx = -5\log x = \log x^{-5}$$

$$\therefore IF = e^{\int P dx} = e^{\log x^{-5}} = x^{5}$$

The solution is

$$v\ IF = \int Q\ IF\ dx + c$$

$$\therefore y^{5}\cdot x^{5} = \int -5x\ x^{5}dx + c$$

$$\therefore y^{-5}\cdot x^{-5} = -5\cdot\frac{x^{-4+1}}{-4+1} + c$$

i.e., $\dfrac{1}{y^5x^5} = \dfrac{5}{3x^3} + c$

Solve $\dfrac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

Sol: Given equation can be written as

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + x\frac{2\sin y\cos y}{\cos^2 y} = x^3$$

$$\therefore \sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \longrightarrow (1)$$

Let $\tan y = v \longrightarrow (2)$

so that $\sec^2 y\dfrac{dy}{dx} = \dfrac{dv}{dx} \longrightarrow (3)$

Eliminate y, by substituting (2),(3) in (1)

$\dfrac{dv}{dx} + 2xv = x^3$ which is linear.

Here $P = 2x,\ Q = x^3$

$$\therefore \int P dx = \int 2x dx = x^2$$

$$\therefore IF = e^{\int P dx} = e^{x^2}$$

The solution is

$$v\ IF = \int Q\ IF\ dx + c$$

$$\therefore \tan y\ e^{x^2} = \int x^3 e^{x^2} dx + c$$

Let $x^2 = t$ so that $2x dx = dt$

$$\therefore \int x^3 e^{x^2} dx = \int t e^t \tfrac{1}{2} dt$$

$$= \frac{1}{2}\left[t\ e^t - 1\ e^t \right]$$

$$= \frac{1}{2} e^t\ t - 1$$

solution is

$$e^{x^2} \tan y = \frac{1}{2} e^{x^2}\ x^2 - 1\ + c$$

8) Factorise in D form

D is d by dx operator

8.1 >

Solve $(D^2 + 14D - 32)\, y = 0$

$D^2 + 14D - 32$ = 0 factorize and get D = 2 and -16

so dy / dx = 2 or dy/dx = -16

Unequal roots gives $y = C_1 e^{2x} + C_2 e^{-16x}$

If Equal roots then $(C_1 + C_2)\, e^{m_1 x}$ = y where m1 is the root

8.2> Solve $d^2y/dx^2 + dy/dx + y = 0$
=> (D^2 + D + 1)y = 0

Gives Imaginary roots so $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$
Where root is α + iβ , α - iβ form

8.3 >
Solve $x \cdot d^2y/dx^2 + 2x \cdot dy/dx - 2y = 0$

Put $y = x^m$
We get $m(m-1) + 2(m-1) = 0$
$(m+2)(m-1) = 0$
m = -2 and m=1
So y = $C_1 x + C_2 x^{-2}$

To recall standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\|x\|$	$\frac{g'(x)}{g(x)}$	$\ln\|g(x)\|$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\|\cos x\|$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\|\tan\frac{x}{2}\right\|$	$\operatorname{cosech} x$	$\ln\left\|\tanh\frac{x}{2}\right\|$
$\sec x$	$\ln\|\sec x + \tan x\|$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\|\sin x\|$	$\coth x$	$\ln\|\sinh x\|$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| \ (0<\|x\|<a)$
	$(a > 0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| \ (\|x\| > a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\|\frac{x+\sqrt{a^2+x^2}}{a}\right\| \ (a > 0)$
	$(-a < x < a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\|\frac{x+\sqrt{x^2-a^2}}{a}\right\| \ (x>a>0)$
$\sqrt{a^2 - x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left. + \frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{x^2-a^2}}{a^2}\right]$

Clairaut's differential equation

IIT JEE 1999

Solve

$\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0$ is

(a) $y = 2$ (b) $y = 2x$

(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

Solution—Given equation is of Clairaut's form

put $\frac{dy}{dx} = p$. Then equation is

$$y = px - p^2$$

Put $p = c$, we find

$y = cx - c^2$, where c is any constant

choose $c = 2$

$$y = 2x - 4$$

-

Clairaut's Equation :

The differential equation of the form

$$y = px + f(p);$$

is called Clairaut's equation. Its primitive is

$$y = Cx + f(C)$$

and is obtained simply by replacing p by C in the given equation.

Example : Solve $p^4 - (x + 2y + 1)\,p^3 + (x + 2y + 2xy)\,p^2 - 2xyp = 0$

or $$p(p-1)(p-x)(p-2y) = 0.$$

The solutions of the component equations of first order and first degree.

Sol. : $\frac{dy}{dx} = 0$, $\frac{dy}{dx} = 1$, $\frac{dy}{dx} - x = 0$, $\frac{dy}{dx} - 2y = 0$

are respectively $y - C = 0$, $y - x - C = 0$, $2y - x^2 - C = 0$, $y - C\,e^{2x} = 0$,

The primitive of the given equation is

$$(y - C)(y - x - C)(2y - x^2 - C)(y - Ce^{2x}) = 0.$$

More Examples

Question 1

Given an Equation (dy/dx) (y + x – 1) – $(dy/dx)^2$ x = y

Take dy/dx = p

The equation is

$$y - px = \frac{p}{p-1}$$

or,
$$y = px + \frac{p}{p-1}$$

It is of Clairaut's form, hence its solution is

$$y = cx + \frac{c}{c-1}$$

Question 2

Given an equation $(dy/dx)^2 (x^2 - a^2) - 2xy(dy/dx) + y^2 - b^2 =0$

Take $dy/dx = p$

So

The given equation is

$$p^2(x^2-a^2)-2pxy+y^2-b^2=0$$

or, $$y^2-2pxy+p^2x^2=a^2p^2+b^2$$

or, $$(y-px)^2=a^2p^2+b^2$$

or, $$y-px=\pm\sqrt{a^2p^2+b^2}$$

or, $$y=px\pm\sqrt{a^2p^2+b^2}$$

Both the component equations are of Clairaut's form

$\therefore$ The solution is

$$y=cx\pm\sqrt{a^2c^2+b^2}$$

or, $$(y-cx)^2=a^2c^2+b^2$$

Question 3

Given an equation (x – a) (dy / dx)^2 + (dy/dx) x = (1 + (dy / dx))y

The given equation is

$$(x-a)p^2 + px = (1+p)y$$

or, $$(1+p)y = px(p+1) - ap^2$$

or, $$y = px - \frac{ap^2}{p+1}$$

which is of Clairaut's form and hence its solutioin is

$$y = cx - \frac{ac^2}{c+1}$$

Question 4

Solve (dy/dx)^2 x^2 – 2 (dy/dx)^2 x + 2py – 2pxy – px + 2p + y^2 + y =0

So question is what is Clairaut's Differential Equations ?

The given equation is

$$p^2x^2 - 2p^2x + 2py - 2pxy - px + 2p + y^2 + y = 0$$

or, $$(y^2 - 2pxy + p^2x^2) + 2p(y - px) + (y - px) + 2p = 0$$

or, $$(y - px)^2 + (2p+1)(y - px) + 2p = 0$$

or, $$(y - px + 2p)(y - px + 1) = 0$$

Both the component equations are of Clairaut's form and hence the solution is

$$(y - cx + 2c)(y - cx + 1) = 0$$

Another Example

Solve $(1 + x^2)\frac{dy}{dx} + xy = x^3y^3$.

(BIHAR CEE 1999)

Solution—$(1 + x^2)\frac{dy}{dx} + xy = x^3y^3$

$$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{1+x^2}\cdot\frac{1}{y^3} = \frac{x^3}{1+x^2}$$

Put $\quad \frac{1}{y^2} = v, \; \frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \quad \frac{1}{-2}\frac{dv}{dx} + \frac{x}{1+x^2}v = \frac{x^3}{1+x^2}$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{2x}{1+x^2}v = \frac{-2x^3}{1+x^2}$$

Now I.F $= e^{\int \frac{-2x}{1+x^2}dx} = e^{-\log(1+x^2)} = \frac{1}{1+x^2}$

The solution is

$$v \cdot \frac{1}{(1+x^2)} = -2\int \frac{x^3}{(1+x^2)^2}dx + c$$

Put $\quad 1 + x^2 = t, \quad 2xdx = dt$

$$\therefore \quad \frac{v}{1+x^2} = -\int \frac{(t-1)}{t^2}dt + c$$

$$\frac{v}{1+x^2} = -\int\left[\frac{1}{t} - \frac{1}{t^2}\right]dt + c$$

$$= -\left[\log t + \frac{1}{t}\right] + c$$

$$\Rightarrow \quad \frac{v}{1+x^2} = -\left[\log(1+x^2) + \frac{1}{1+x^2}\right]c$$

Returning to y

$$\frac{1}{y^2(1+x^2)} = -\left[\log(1+x^2) + \frac{1}{1+x^2}\right] + c.$$

More examples teaches us better

The order of a differential equation is the order of the highest derivative included in the equation.

Find the order of the following Differential Equations

1. $\frac{dy}{dx} + y^2 x = 2x$
2. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$
3. $10y'' - y = e^x$
4. $\frac{d^3y}{dx^3} - x\frac{dy}{dx} + (1-x)y = \sin y$

1. The highest derivative is dy/dx, the first derivative of y. The order is therefore 1

2. The highest derivative is d^2y / dx^2, a second derivative. The order is therefore 2

3. The highest derivative is the second derivative y". The order is 2

4. The highest derivative is the third derivative d^3 / dy^3. The order is 3

Another example

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 = e^t$$

The highest order differential coefficient is $\frac{d^3x}{dt^3}$ and its power is 1.

So, it is a non-linear differential equation with order 3 and degree 1.

Another example

$$\left(\frac{dy}{dx}\right)^2 + \frac{1}{\left(\frac{dy}{dx}\right)} = 2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 + 1 = 2\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 - 2\left(\frac{dy}{dx}\right) + 1 = 0$$

This is a polynomial in $\frac{dy}{dx}$.

The highest order differential coefficient is $\frac{dy}{dx}$ and its power is 3.

So, it is a non-linear differential equation with order 1 and degree 3.

Which of these differential equations are linear?

1. $\frac{dy}{dx} + x^2 y = x$
2. $\frac{1}{x}\frac{d^2y}{dx^2} - y^3 = 3x$
3. $\frac{dy}{dx} - \ln y = 0$
4. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2\sin x$

1. Both dy/dx and y are linear. The differential equation is linear

2. The term y^3 is not linear. The differential equation is not linear

3. The term ln y is not linear. This differential equation is not linear

4. The terms d^3y / dx^3, d^2y / dx^2 and dy / dx are all linear. The differential equation is linear

Determine the order and state the linearity of each differential below

1. $(\frac{d^3y}{dx^3})^4 + 2\frac{dy}{dx} = \sin x$
2. $\frac{dy}{dx} - 2xy = x^2 - x$
3. $\frac{dy}{dx} - \sin y = -x$
4. $\frac{d^2y}{dx} = 2xy$

1. order 3 , non linear

2. order 1 , linear

3. order 1 , non linear

4. order 2 , linear

Example of a variable separable type

Solve $3e^x \tan y dx + (1+e^x)\sec^2 y dy = 0$ given $y = \frac{\pi}{4}$ when $x = 0$.

Sol: Given $3e^x \tan y dx + (1+e^x)\sec^2 y dy = 0$

Dividing with $\tan y \cdot (1+e^x)$, we get

$$\frac{3e^x}{1+e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

$$\therefore \int \frac{3e^x}{1+e^x}dx + \int \frac{\sec^2 y}{\tan y}dy = c$$

$$\therefore 3\log(1+e^x) + \log(\tan y) = \log C$$

$$\therefore (1+e^x)^3 \tan y = C$$

Put $x = 0$ and $y = \frac{\pi}{4}$

$$(1+e^0)^3 \tan\frac{\pi}{4} = C \Rightarrow C = 8$$

$\therefore$ The required solution is

$$(1+e^x)^3 \tan y = 8$$

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Good Luck to you for your Preparations, References, and Exams

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