Spoon Feeding Definite Integrals

Simplified Knowledge Management Classes Bangalore

My name is Subhashish Chattopadhyay. I have been teaching for IIT-JEE, Various International Exams (such as IMO [International Mathematics Olympiad], IPhO [International Physics Olympiad], IChO [International Chemistry Olympiad]), IGCSE (IB), CBSE, I.Sc, Indian State Board exams such as WB-Board, Karnataka PU-II etc since 1989. As I write this book in 2016, it is my 25 th year of teaching. I was a Visiting Professor to BARC Mankhurd, Chembur, Mumbai, Homi Bhabha Centre for Science Education (HBCSE) Physics Olympics camp BARC Campus.

I am Life Member of …

- IAPT (Indian Association of Physics Teachers)

- IPA (Indian Physics Association)

- AMTI (Association of Mathematics Teachers of India)

- National Human Rights Association

- Men's Rights Movement (India and International)

- MGTOW Movement (India and International)

And also of

IACT (Indian Association of Chemistry Teachers)

The selection for National Camp (for Official Science Olympiads - Physics, Chemistry, Biology, Astronomy) happens in the following steps ….

1) **NSEP** (National Standard Exam in Physics) and **NSEC** (National Standard Exam in Chemistry) held around 24 rth November. Approx 35,000 students appear for these exams every year. The exam fees is Rs 100 each. Since 1998 the IIT JEE toppers have been topping these exams and they get to know their rank / performance ahead of others.

2) **INPhO** (Indian National Physics Olympiad) and **INChO** (Indian National Chemistry Olympiad). Around 300 students in each subject are allowed to take these exams. Students coming from outside cities are paid fair from the Govt of India.

3) The Top 35 students of each subject are invited at HBCSE (Homi Bhabha Center for Science Education) Mankhurd, near Chembur, BARC, Mumbai. After a 2-3 weeks camp the top 5 are selected to represent India. The flight tickets and many other expenses are taken care by Govt of India.

Since last 50 years there has been no dearth of "Good Books". Those who are interested in studies have been always doing well. This e-Book does not intend to replace any standard text book. These topics are very old and already standardized.

There are 3 kinds of Text Books

- The thin Books - Good students who want more details are not happy with these. Average students who need more examples are not happy with these. Most students who want to "Cram" quickly and pass somehow find the thin books "good" as they have to read less !!

- The Thick Books - Most students do not like these, as they want to read as less as possible. Average students are "busy" with many other things and have no time to read all these.

- The Average sized Books - Good students do not get all details in any one book. Most bad students do not want to read books of "this much thickness" also !!

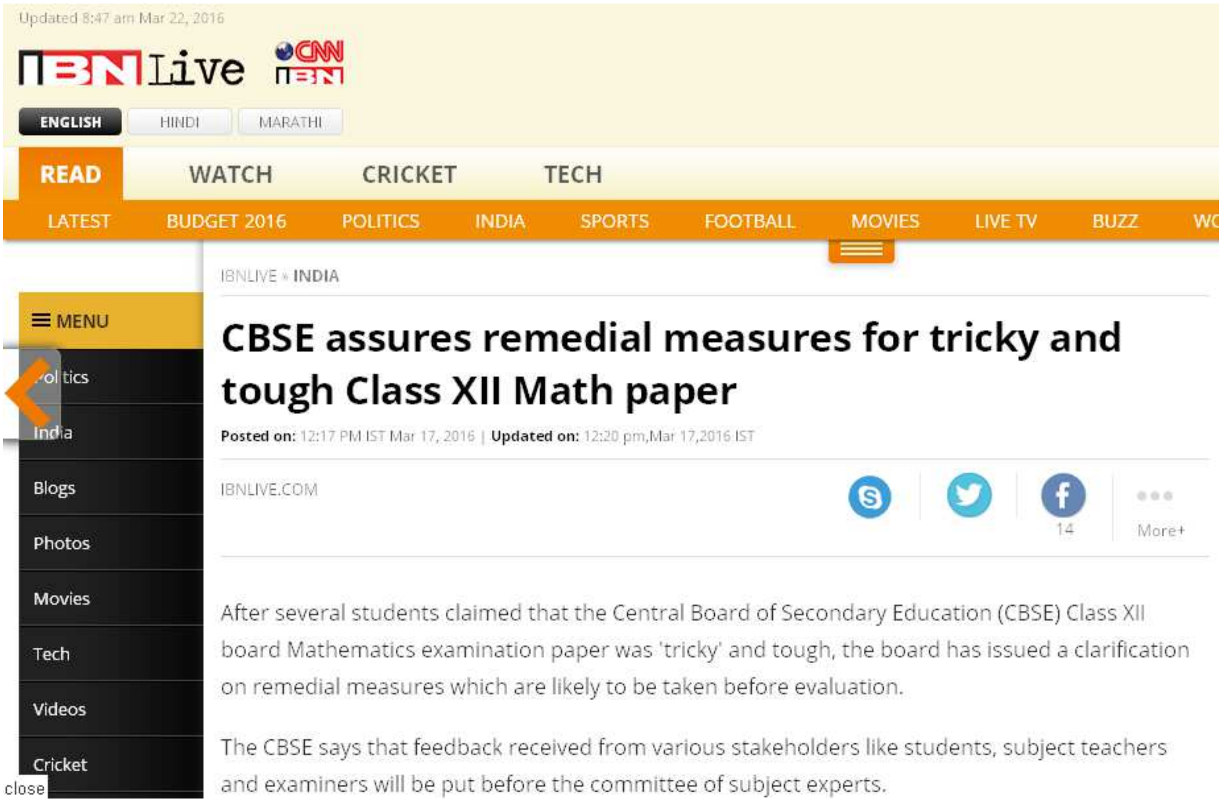
We know there can be no shoe that's fits in all.

Printed books are not e-Books! Can't be downloaded and kept in hard-disc for reading "later" ....

So if you read this book later, you will get all kinds of examples in a single place. This becomes a very good "Reference Material". I sincerely wish that all find this "very useful".

Students who do not practice lots of problems, do not do well. The rules of "doing well" had never changed …. Will never change !

After 2016 CBSE Mathematics exam, lots of students complained that the paper was tough!



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CBSE assures remedial measures for tricky and tough Class XII Math paper

Posted on: 12:17 PM IST Mar 17, 2016 | **Updated on:** 12:20 pm,Mar 17,2016 IST

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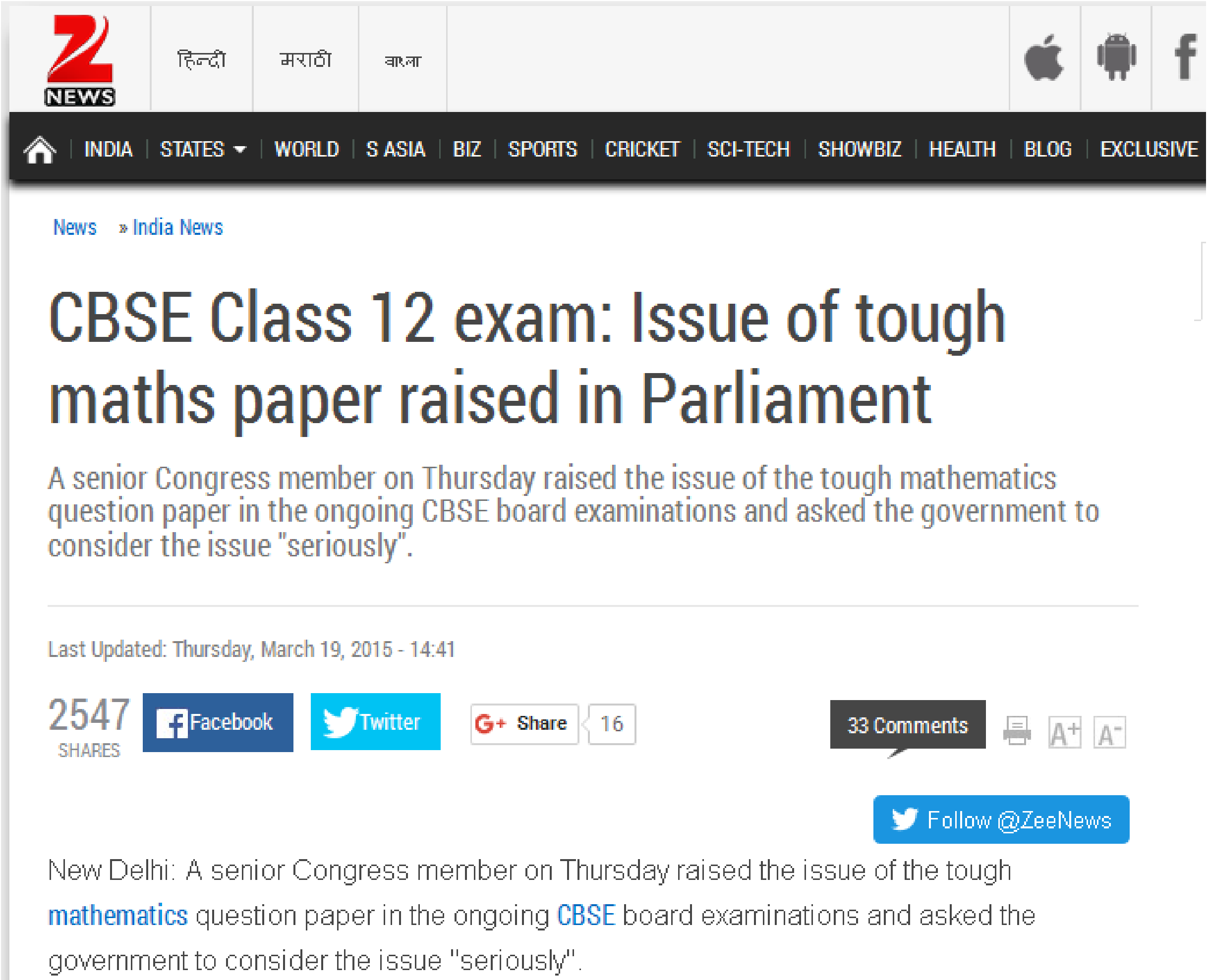
After several students claimed that the Central Board of Secondary Education (CBSE) Class XII board Mathematics examination paper was 'tricky' and tough, the board has issued a clarification on remedial measures which are likely to be taken before evaluation.

The CBSE says that feedback received from various stakeholders like students, subject teachers and examiners will be put before the committee of subject experts.

In 2015 also the same complain was there by many students



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News » India News

CBSE Class 12 exam: Issue of tough maths paper raised in Parliament

A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

Last Updated: Thursday, March 19, 2015 - 14:41

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New Delhi: A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

In March 2016, students of Karnataka PU-II also complained the same, regarding standard 12 (PU-II Mathematics Exam). Even though the Math Paper was identical to previous year, most students had not even solved the 2015 Question Paper.


Friday, March 25, 2016 - 13:28

The NEWS Minute

HOME NEWS ANDHRA KARNATAKA KERALA TAMIL NADU TELANGANA CULTURE MEDIA BLOG

Exams

Online petition for lenient evaluation of K'taka II PU math paper gets over 8000 supporters

The campaign, which was launched on Monday, has garnered over 8000 supporters

TNM Staff | Wednesday, March 16, 2016 - 09:32

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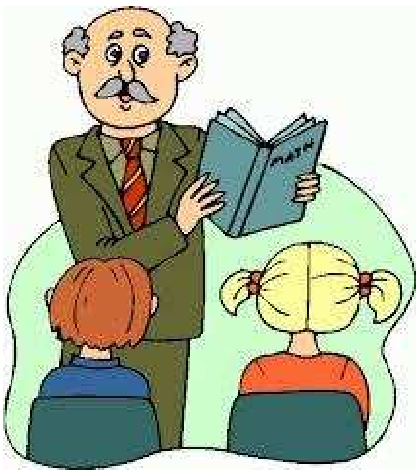
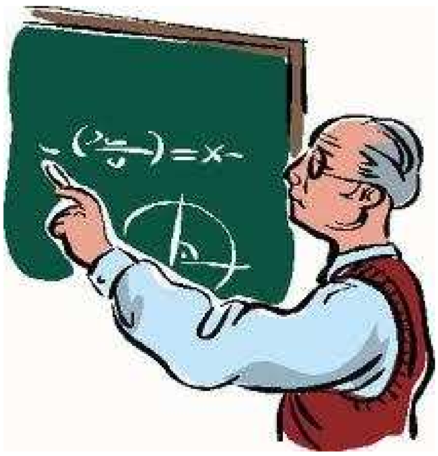
Following a "very tough" math paper that left many II PU students in tears, Saket Ravindran a student launched an online campaign demanding lenient evaluation.

These complains are not new. In fact since last 40 years, (since my childhood), I always see this; every year the same setback, same complain!

In this e-Book I am trying to solve this problem. Those students who practice can learn.

No one can help those who are not studying, or practicing.





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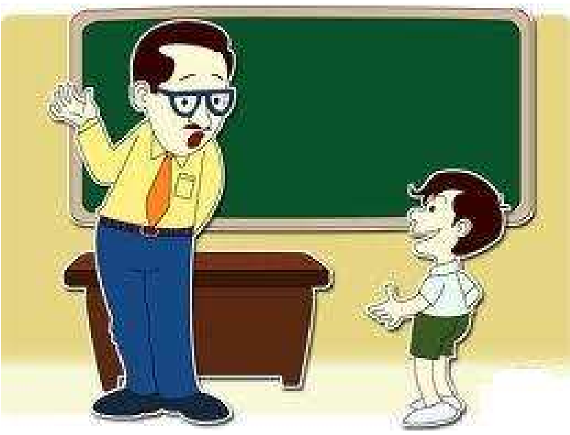
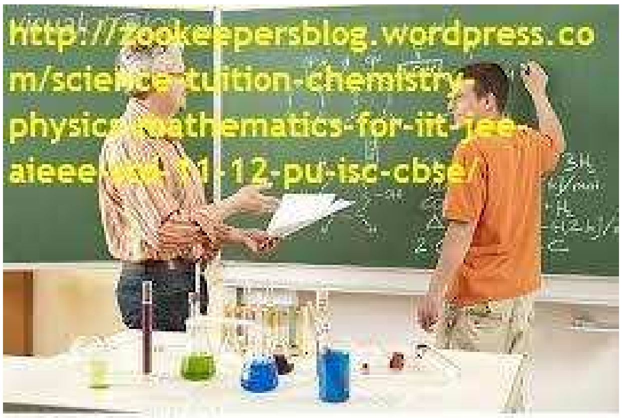




A very polite request :

I wish these e-Books are read only by Boys and Men. Girls and Women, better read something else; learn from somewhere else.

Preface

We all know that in the species "Homo Sapiens ", males are bigger than females. The reasons are explained in standard 10, or 11 (high school) Biology texts. **This shapes or size, influences all of our culture**. Before we recall / understand the reasons once again, let us see some random examples of the influence

Random - 1

If there is a Road rage, then who all fight ? (generally ?). Imagine two cars driven by adult drivers. Each car has a woman of similar age as that of the Man. The cars " touch " or " some issue happens". Who all comes out and fights ? Who all are most probable to drive the cars ?

(Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win)

Random - 2

Heavy metal music artists are all Men. Metallica, Black Sabbath, Motley Crue, Megadeth, Motorhead, AC/DC, Deep Purple, Slayer, Guns & Roses, Led Zeppelin, Aerosmith ….. the list can be in thousands. All these are grown-up Boys, known as Men.

(Men strive for perfection. Men are eager to excel. Men work hard. Men want to win.)

Random - 3

Apart from Marie Curie, only one more woman got Nobel Prize in Physics. (Maria Goeppert Mayer - 1963). So, … almost all are men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 4

The best Tabla Players are all Men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 5

History is all about, which Kings ruled. Kings, their men, and Soldiers went for wars. History is all about wars, fights, and killings by men.

Boys start fighting from school days. Girls do not fight like this

(Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win.)

Random - 6

The highest award in Mathematics, the " Fields Medal " is around since decades. Till date only one woman could get that. (Maryam Mirzakhani - 2014). So, … almost all are men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 7

Actor is a gender neutral word. Could the movie like " Top Gun " be made with Female actors ? The best pilots, astronauts, Fighters are all Men.

Random - 8

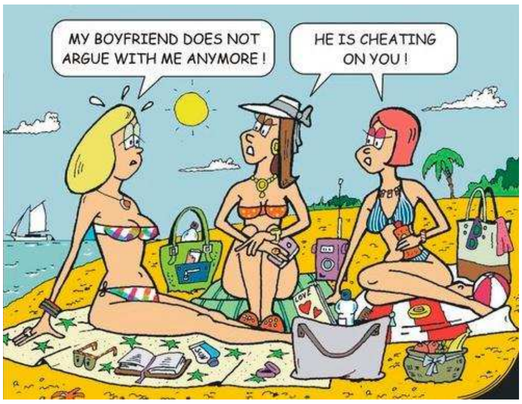
In my childhood had seen a movie named “ The Tower in Inferno “. In the movie when the tall tower is in fire, women were being saved first, as only one lift was working….

Many decades later another movie is made. A box office hit. “ The Titanic “. In this also …. As the ship is sinking women are being saved. **Men are disposable**. Men may get their turn later…

Movies are not training programs. Movies do not teach people what to do, or not to do. Movies only reflect the prevalent culture. Men are disposable, is the culture in the society. Knowingly, unknowingly, the culture is depicted in Movies, Theaters, Stories, Poems, Rituals, etc. I or you can't write a story, or make a movie in which after a minor car accident the Male passengers keep seating in the back seat, while the both the women drivers come out of the car and start fighting very bitterly on the road. There has been no story in this world, or no movie made, where after an accident or calamity, Men are being helped for safety first, and women are told to wait.

Random - 9

Artists generally follow the prevalent culture of the Society. In paintings, sculptures, stories, poems, movies, cartoon, Caricatures, knowingly / unknowingly, " the prevalent Reality " is depicted. The opposite will not go well with people. If deliberately " the opposite " is shown then it may only become a special art, considered as a special mockery.



Random - 10

Men go to "girl / woman's house" to marry / win, and bring her to his home. That is a sort of winning her. When a boy gets a " Girl-Friend ", generally he and his friends consider that as an achievement. The boy who " got / won " a girl-friend feels proud. His male friends feel, jealous, competitive and envious. Millions of stories have been written on these themes. Lakhs of movies show this. Boys / Men go for " bike race ", or say " Car Race ", where the winner " gets " the most beautiful girl of the college.




(Men want to excel. Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win.)

Prithviraj Chauhan ' went ` to " pickup " or " abduct " or " win " or " bring " his love. There was a Hindi movie (hit) song … " Pasand ho jaye, to ghar se utha laye ". It is not other way round. Girls do not go to Boy's house or man's house to marry. Nor the girls go in a gang to " pick-up " the boy / man and bring him to their home / place / den.

Random - 11

Rich people; often are very hard working. Successful business men, establish their business (empire), amass lot of wealth, with lot of difficulty. Lots of sacrifice, lots of hard work, gets into this. Rich people's wives had no contribution in this wealth creation. Women are smart, and successful upto the extent to choose the right/rich man to marry. So generally what happens in case of Divorces ? Search the net on " most costly divorces " and you will know. The women;(who had no contribution at all, in setting up the business / empire), often gets in Billions, or several Millions in divorce settlements.


Number 1

Rupert & Anna Murdoch -- $1.7 billion

One of the richest men in the world, Rupert Murdoch developed his worldwide media empire when he inherited his father's Australian newspaper in 1952. He married Anna Murdoch in the '60s and they remained together for 32 years, springing off three children.

They split amicably in 1998 but soon Rupert forced Anna off the board of News Corp and the gloves came off. The divorce was finalized in June 1999 when Rupert agreed to let his ex-wife leave with $1.7 billion worth of his assets, $110 million of it in cash. Seventeen days later, Rupert married Wendi Deng, one of his employees.

Ted Danson & Casey Coates -- $30 million

Ted Danson's claim to fame is undoubtedly his decade-long stint as Sam Malone on NBC's celebrated sitcom Cheers . While he did other TV shows and movies, he will always be known as the bartender of that place where everybody knows your name. He met his future first bride Casey, a designer, in 1976 while doing Erhard Seminars Training.

Ten years his senior, she suffered a paralyzing stroke while giving birth to their first child in 1979. In order to nurse her back to health, Danson took a break from acting for six months. But after two children and 15 years of marriage, the infatuation fell to pieces. Danson had started seeing Whoopi Goldberg while filming the comedy, Made in America and this precipitated the 1992 divorce. Casey got $30 million for her trouble.

See https://zookeepersblog.wordpress.com/misandry-and-men-issues-a-short-summary-at-single-place/

See http://skmclasses.kinja.com/save-the-male-1761788732

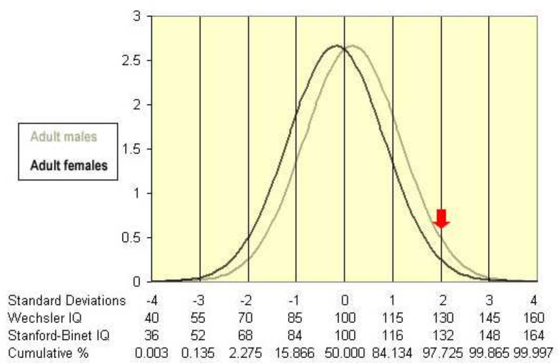
It was Boys and Men, who brought the girls / women home. The Laws are biased, completely favoring women. The men are paying for their own mistakes.

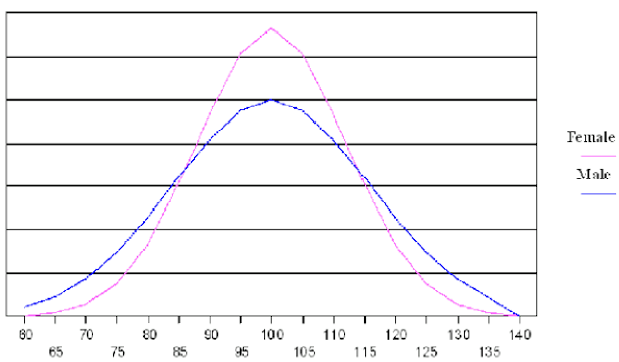
See https://zookeepersblog.wordpress.com/biased-laws/

(Man brings the Woman home. When she leaves, takes away her share of big fortune!)

Random - 12

A standardized test of Intelligence will never be possible. It never happened before, nor ever will happen in future; where the IQ test results will be acceptable by all. In the net there are thousands of charts which show that the intelligence scores of girls / women are lesser. Debates of Trillion words, does not improve performance of Girls.





I am not wasting a single second debating or discussing with anyone, on this. I am simply accepting ALL the results. IQ is only one of the variables which is required for success in life. Thousands of books have been written on " Networking Skills ", EQ (Emotional Quotient), Drive, Dedication, Focus, " Tenacity towards the end goal " … etc. In each criteria, and in all together, women (in general) do far worse than men. Bangalore is known as " ….. capital of India ". [Fill in the blanks]. The blanks are generally filled as " Software Capital ", " IT Capital ", " Startup Capital ", etc. I am member in several startup eco-systems / groups. I have attended hundreds of meetings, regarding " technology startups ", or " idea startups ". These meetings have very few women. Starting up new companies are all " Men's Game " / " Men's business ". Only in Divorce settlements women will take their goodies, due to Biased laws. There is no dedication, towards wealth creation, by women.

Random - 13

Many men, as fathers, very unfortunately treat their daughters as " Princess ". Every " non-performing " woman / wife was " princess daughter " of some loving father. Pampering the girls, in name of " equal opportunity ", or " women empowerment ", have led to nothing.

See http://skmclasses.kinja.com/progressively-daughters-become-monsters-1764484338

See http://skmclasses.kinja.com/vivacious-vixens-1764483974

There can be thousands of more such random examples, where " Bigger Shape / size " of males have influenced our culture, our Society. **Let us recall the reasons**, that we already learned in standard 10 - 11, Biology text Books. In humans, women have a long gestation period, and also spends many years (almost a decade) to grow, nourish, and stabilize the child. (Million years of habit) Due to survival instinct Males want to inseminate. Boys and Men fight for the " facility (of womb + care) " the girl / woman may provide. Bigger size for males, has a winning advantage. Whoever wins, gets the " woman / facility ". The male who is of " Bigger Size ", has an advantage to win.... Leading to Natural selection over millions of years. In general " Bigger Males "; the " fighting instinct " in men; have led to wars, and solving tough problems (Mathematics, Physics, Technology, startups of new businesses, Wealth creation, Unreasonable attempts to make things [such as planes], Hard work ....)

So let us see the IIT-JEE results of girls. Statistics of several years show that there are around 17, (or less than 20) girls in top 1000 ranks, at all India level. Some people will yet not understand the performance, till it is said that ... year after year we have around 980 boys in top 1000 ranks. Generally we see only 4 to 5 girls in top 500. In last 50 years not once any girl topped in IIT-JEE advanced. Forget about Single digit ranks, double digit ranks by girls have been extremely rare. It is all about " good boys ", " hard working ", " focused ", "**Bel-esprit** " **boys**.

In 2015, Only 2.6% of total candidates who qualified are girls (upto around 12,000 rank). while 20% of the Boys, amongst all candidates qualified. The Total number of students who appeared for the exam were around 1.4 million for IIT-JEE main. Subsequently 1.2 lakh (around 120 thousands) appeared for IIT-JEE advanced.

IIT-JEE results and analysis, of many years is given at https://zookeepersblog.wordpress.com/iit-jee-iseet-main-and-advanced-results/

In Bangalore it is rare to see a girl with rank better than 1000 in IIT-JEE advanced. We hardly see 6-7 boys with rank better than 1000. Hardly 2-3 boys get a rank better than 500.

See http://skmclasses.weebly.com/everybody-knows-so-you-should-also-know.html

Professor Subhashish Chattopadhyay

-

Spoon Feeding Series - Definite Integrals

Recall the various tricks, formulae, and rules of solving Indefinite Integrals

(i) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(ii) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$

(iii) $\int \frac{dx}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C$

(iv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C = \cosh^{-1}\left(\frac{x}{a}\right) + C$

(vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$

(vii) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log|x+\sqrt{x^2+a^2}|\right] + C$

(viii) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + C$

(ix) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2} - a^2\log|x+\sqrt{x^2-a^2}|\right] + C$

(x) $\int (px+q)\sqrt{ax^2+bx+c}\,dx = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \left(\frac{q-pb}{2a}\right)\int \sqrt{ax^2+bx+c}\,dx$

- $\int e^x dx = e^x$
- $\int e^{ax} dx = \frac{1}{a} e^{ax}$
- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$
- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$
- $\int a^x dx = \frac{a^x}{lna} + c$

- $\int \log x dx = x(\log x - 1) + c$
- $\int \frac{1}{x} dx = \log |x| + c$
- $\int a^x dx = a^x \log x + c$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$

- $\int \csc x \cot x dx = -\csc x + c$
- $\int \csc^2 x dx = -\cot x + c$
- $\int \sec x \tan x dx = \sec x + c$
- $\int \sec^2 x dx = \tan x + c$
- $\int \sin x dx$ = - \cos x + c$
- $\int \cos x dx = \sin x + c$

- $\int (ax+b)^n = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$, $n \neq 1
- $\int \frac{dx}{(ax+b)} = \frac{1}{a} \log |ax+b| + C$
- $\int e^{ax+b} = \frac{1}{a} e^{ax+b} + C$
- $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$
- $\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + C$
- $\int \csc^2(ax+b) dx = \frac{-1}{a} \cot(ax+b) + C$
- $\int \csc(ax+b) \cot(ax+b) dx = \frac{-1}{a} \csc(ax+b) + C$

For Integrals of the form

(i) $\int \frac{dx}{a + b \sin x}$ (ii) $\int \frac{dx}{a + b \cos x}$ (iii) $\int \frac{dx}{a \sin x + b \cos x + c}$

Put $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$, $\sin x - \frac{2 \tan x/2}{1 + \tan^2 x/2}$

Some advanced procedures….

Integral	Substitution
$\int \frac{x^m}{(a+bx)^p}\,dx$, m is a + ve integer	Put $a + bx = z$
$\int \frac{dx}{x^m (a+bx)^p}$, where either ($m$ and p positive integers) or (m and p are fractions, but $m + p$ = integers > 1)	Put $a + bx = zx$
$\int x^m (a+bx^n)^p\,dx$, where m, n, p are rationals.	
(i) p is a + ve integer	Apply Binomial theorem to $(a + bx^n)^p$
(ii) p is a − ve integer	Put $x = z^k$ where k = common denominator of m and n.
(iii) $\frac{m+1}{n}$ is an integer	Put $(a + bx^n) = z^k$ where k = denominator of p.
(iv) $\frac{m+1}{n} + p$ is an integer	Put $a + bx^n = x^n z^k$ where k = denominator of fraction p.

$$\int \frac{x^2\,dx}{x^4 + k x^2 + a^4} = \frac{1}{2}\int \frac{(x^2+a^2)\,dx}{(x^4 + k x^2 + x^4)} + \frac{1}{2}\int \frac{(x^2-a^2)\,dx}{(x^4 + k x^2 + a^4)}$$

$$\int \frac{dx}{(x^4 + k x^2 + a^4)} = \frac{1}{2a^2}\int \frac{(x^2+a^2)\,dx}{(x^4 + k x^2 + a^4)} - \frac{1}{2a^2}\int \frac{(x^2-a^2)\,dx}{(x^4 + k x^2 + a^4)}$$

$$\int \frac{dx}{(x^2+k)^n} = \frac{x}{k(2n-2)(x^2+k)^{n-1}} + \frac{(2n-3)}{k(2n-2)}\int \frac{dx}{(x^2+k)^{n-1}}$$

For $\int \frac{dx}{(Ax^2+Bx+C)\sqrt{(ax^2+bx+c)}}$ we need to substitute $\frac{ax^2+bx+c}{Ax^2+Bx+C} = t^2$

Every student knows that the last step is …

$$\int_a^b f(x)\,dx = [F(x) + c]_a^b = F(b) - F(a)$$

Definite Integrals have to be solved by (more than) 14 different ways, depending on the type of problem.

Type 1 - Here no property, specific to Definite Integrals is used.

The integration is solved completely as Indefinite. Finally the Upper and Lower limits are substituted.

Example - 1.1 -

If we need to solve $\int_0^{\pi} \frac{1}{1+\sin x} dx$

we should know how to integrate $\int \frac{1}{1+\sin x} dx$ (Indefinite Integral)

In the solution, notice that no special or specific property of Definite Integral is being used.

Multiplying Numerator and Denominator by $(1 - \sin x)$

$$I = \int_0^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$
$$= \int_0^{\pi} \frac{(1-\sin x)}{(1^2 - \sin^2 x)} dx$$
$$= \int_0^{\pi} \frac{1-\sin x}{(\cos^2 x)} dx$$
$$= \int_0^{\pi} \frac{1}{\cos^2 x} dx - \int_0^{\pi} \frac{\sin x}{\cos^2 x} dx$$
$$= \int_0^{\pi} \sec^2 x dx - \int_0^{\pi} \tan x . \sec x dx$$
$$= [\tan x]_0^{\pi} - [\sec x]_0^{\pi}$$
$$= [\tan \pi - \tan 0] - [\sec \pi - \sec 0]$$
$$= [0 - 0] - [-1 - 1]$$
$$= 2$$

Similarly **example - 1.2 -**

$$\int_1^2 \log x dx = \Big[x \log x - x \Big]_1^2$$

As we know from indefinite integrals that Integration of Ln |x| is x Ln | x | - x

If we substitute the upper limit we get 2 ln 2 - 2

And substituting the lower limit we get 1 ln 1 - 1 = -1

So final result is 2 ln 2 - 2 - (-1) = 2 ln 2 - 1

Example - 1.3 -

If we need to integrate by parts then do not apply the limits at intermediate steps.

Solve the whole problem as indefinite and then finally apply the limits

Recall $\int uv\ dx = u \int v\ dx - \int \left(u' \int v\ dx \right)\ dx.$

So to solve $\int_0^1 (x^2+1)\, e^{-x}\, dx$ we proceed as above equation

Let $u = x^2 + 1$ and $dv = e^{-x}\, dx$. Then $du = 2x\, dx$ and $v = -e^{-x}$

I = $\int_0^1 (x^2+1)\, e^{-x}\, dx = \left[-(x^2+1)e^{-x}\right]_0^1 + 2\int_0^1 xe^{-x}\, dx$

$$\int_0^1 xe^{-x}\, dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\, dx = \left[-e^{-x}(x+1)\right]_0^1$$

Thus finally the required Solution is $\int_0^1 (x^2+1)\, e^{-x}\, dx = \left[-e^{-x}\left(x^2+2x+3\right)\right]_0^1 = -6e^{-1} + 3$

Example - 1.4 -

Show that $\int_0^1 x \tan^{-1} x dx = \frac{\pi}{4} - \frac{1}{2}$

$$\int_0^1 x \tan^{-1} x\, dx = \tan^{-1} x \int_0^1 x dx - \int_0^1 \left(\int x dx\right) \frac{d}{dx}\left(\tan^{-1} x\right) dx$$

$$= \left[\frac{x^2}{2} \tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2} dx$$

$$= \left[\frac{x^2}{2} \tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2} dx$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left[\int_0^1 dx - \int_0^1 \frac{dx}{1+x^2}\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1} x\right]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[1 - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

Example - 1.5 -

Solve $\int_0^2 x\sqrt{x+2}\,dx$ Put x + 2 = t^2 so dx = 2t dt at x = 0 t = √2 at x = 2 x + 2 = 4 = t^2 => t = 2

$$I = \int_{\sqrt{2}}^{2}(t^2-2)\sqrt{t^2}\,2t\,dt$$
$$= 2\int_{\sqrt{2}}^{2}(t^2-2)t^2\,dt$$
$$= 2\int_{\sqrt{2}}^{2}(t^4-2t^2)\,dt$$
$$= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^{2}$$
$$= 2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$
$$= 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$
$$= 2\left[\frac{16+8\sqrt{2}}{15}\right]$$
$$= \frac{16(2+\sqrt{2})}{15}$$
$$= \frac{16\sqrt{2}(\sqrt{2}+1)}{15}$$

Example - 1.6 -

Solve $\int_{\frac{1}{3}}^{1}\frac{(x-x^3)^{\frac{1}{3}}}{x^4}dx$

let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

When $x = \frac{1}{3}$, $\theta = \sin^{-1}\left(\frac{1}{3}\right)$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta-\sin^3\theta)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta\, d\theta$$
$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(1-\sin^2\theta)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta\, d\theta$$
$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^4\theta}\cos\theta\, d\theta$$
$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^2\theta\sin^2\theta}\cos\theta\, d\theta$$
$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\cos\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}}\text{cosec}^2\theta\, d\theta$$
$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}(\cot\theta)^{\frac{5}{3}}\text{cosec}^2\theta\, d\theta$$

Let $\cot\theta = t \Rightarrow -\text{cosec}2\theta\, d\theta = dt$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

$$\therefore I = -\int_{2\sqrt{2}}^{0}(t)^{\frac{5}{3}}dt$$
$$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$
$$= -\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$
$$= -\frac{3}{8}\left[-(2\sqrt{2})^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$
$$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$
$$= \frac{3}{8}[16]$$
$$= 3 \times 2$$
$$= 6$$

AIEEE (now known as IIT-JEE main) - 2004

Solve

The value of $I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1+\sin 2x}}\,dx$ is

(a) 2 (b) 1 (c) 0 (d) 3

(a) : $\int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{(\sin x + \cos x)^2}}\,dx$

$= \int_0^{\pi/2} (\sin x + \cos x)dx$

$= \left(\frac{\cos x}{-1} + \sin x\right)_0^{\frac{\pi}{2}}$

$= 1 - (-1) = 2$

AIEEE (now known as IIT-JEE main) - 2007

The solution for x of the equation $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $2\sqrt{2}$ (c) 2 (d) π

Solution :

$$\left[\sec^{-1} t\right]_{\sqrt{2}}^{x} = \frac{\pi}{2}$$

$$\sec^{-1} x - \sec^{-1}\sqrt{2} = \frac{\pi}{2} \Rightarrow \sec^{-1} x = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

$x = -\sqrt{2}$ There is no correct option.

Example - 1.7 -

If $I = \int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)}\, dx$, then

I equals

(a) $\frac{1}{2}\log 6 + \frac{1}{10}$ (b) $\frac{1}{2}\log 6 - \frac{1}{10}$

(c) $\frac{1}{2}\log 3 - \frac{1}{10}$ (d) $\frac{1}{2}\log 2 + \frac{1}{10}$

Solution

$$2x^5 + x^4 - 2x^3 + 2x^2 + 1$$
$$= 2x^3(x^2 - 1) + (x^2 + 1)^2$$

$$\therefore\ I = \int_2^3 \frac{2x^3(x^2-1) + (x^2+1)^2}{(x^2+1)^2 + (x^2-1)}\, dx$$

$$= \int_2^3 \frac{2x^3\, dx}{(x^2+1)^2} + \int_2^3 \frac{dx}{x^2-1}$$

$$= I_1 + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\Bigg]_2^3$$

$$= I_1 + \frac{1}{2}\left(\log\frac{1}{2} - \log\frac{1}{3}\right)$$

where $I_1 = \int_2^3 \frac{x^2}{(x^2+1)^2}(2x)\, dx$

Put $x^2 + 1 = t$, $2x\, dx = dt$

$$\therefore\ I_1 = \int_5^{10} \frac{t-1}{t^2}\, dt = \left(\log|t| + \frac{1}{t}\right)\Bigg]_5^{10}$$

$$= \log 2 - \frac{1}{10}$$

Thus, $I = \frac{1}{2}\log 6 - \frac{1}{10}$

-

How to Shift limits in Definite Integrals, when variable is changed is explained at

https://archive.org/details/ShiftingOfLimitsInDefiniteIntegralStepsDiscussed1

-

Type 2 - Here special properties of Definite Integrals are used

Let us see the list of properties

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \text{. In particular, } \int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x) \text{ and}$$
$$0 \text{ if } f(2a-x) = -f(x)$$

(i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function, i.e., if $f(-x) = f(x)$.

(ii) $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.

The property of Modulus

$$\left| \int_a^b f(x)\,dx \right| < \int_a^b |f(x)|\,dx$$

An Example to start the discussion

The absolute value of $\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx$ is

(a) less than 10^{-7} (b) more than 10^{-7}
(c) less than 10^{-6} (d) more than 10^{-6}

Solution

(a, c). $= \left|\int_{10}^{19} \frac{\sin x}{1+x^8} dx\right| \leq \int_{10}^{19} \frac{|\sin x|}{1+x^8} dx$

$\left[\because |f(x)| \leq \int |f(x)| dx\right]$

$\leq \int_{10}^{19} \frac{dx}{1+x^8}$ $\quad [\because |\sin x| \leq 1]$

$< \int_{10}^{19} \frac{dx}{x^8}$ $\quad \left[\because 1+x^8 > x^8 \Rightarrow \frac{1}{1+x^8} < \frac{1}{x^8}\right]$

$< \int_{10}^{19} \frac{dx}{10^8}$ $\quad \left[\because x > 10 \Rightarrow \frac{1}{x} < \frac{1}{10}\right]$

$= \frac{1}{10^8}(19-10)$

$= 9 \times 10^{-8} < 10 \times 10^{-8} < 10^{-7}$

Again, $\because 10^7 > 10^6 \Rightarrow 10^{-7} < 10^{-6}$

$\therefore$ given integral is $< 10^{-6}$

If the function f (x) increases and has a concave graph in the interval [a, b], then

$$(b-a)\, f(a) < \int_a^b f(x)\, dx < (b-a)\, \frac{f(a)+f(b)}{2}$$

If the function f (x) increases and has a convex graph in the interval [a, b], then

$$(b-a)\,\frac{f(a)+f(b)}{2} < \int_a^b f(x)\,dx < (b-a)\,f(b)$$

Example - 2.1 - Solve $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

As indefinite integral when we solve this we express Cos^2 x as Cos 2x form

But with limits 0 to π/2 we better use $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx \quad \text{- (1)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2}-x\right)dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx \quad \text{- (2)}$$

Adding (1) and (2) we get

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\,dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Example - 2.2 - Is one of the most common questions, asked Lakhs of times in all sorts of school and entrance exams.

Find $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ Modification of this problem is to divide the Denominator by √Sin x bringing the numerator down (below Denominator). So the denominator becomes 1 + √Cot x

Also the problem could have been $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos}}{\sqrt{\cos}+\sqrt{\sin x}}dx$ or without roots $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}dx$

Or $\int_0^{\frac{\pi}{2}} \frac{\text{Cos } x}{\sin x + \cos x}dx$ The approach to solve these remain the same

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad - (1)$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos}}{\sqrt{\cos}+\sqrt{\sin x}}dx \quad - (2)$$

Adding (1) and (2), we obtain

->

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Example - 2.3 - Not only Sin x or √Sin x but $\text{Sin}^{3/2}$ x or $\text{Sin}^{5/2}$ x or $\text{Sin}^{(2N+1)/2}$ x

meaning Cos or $\text{Sin}^{(\text{Odd Natural Number})/2}$ x will have the same approach

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx \quad - (1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + \cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx \quad - (2)$$

Adding (1) and (2), we obtain ->

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Spoon feeding

If $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$, then I equals

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

Ans. (a)

Solution We can write

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad (1)$$

Using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx$, we can write

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Adding (1) and (2), we get

$$2I = \int_{\pi/6}^{\pi/3} dx = x]_{\pi/6}^{\pi/3} = \frac{\pi}{6}$$

$$\Rightarrow I = \pi/12$$

Example - 2.4 -

Solve $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log(2\sin x \cos x)\}\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log \sin x - \log \cos x - \log 2\}\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \sin x - \log \cos x - \log 2\}\, dx \quad - (1)$$

Applying $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \cos x - \log \sin x - \log 2\}\, dx \quad (2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2)\, dx$$

$$\Rightarrow 2I = -2\log 2 \int_0^{\frac{\pi}{2}} 1\, dx$$

$$\Rightarrow I = -\log 2 \left[\frac{\pi}{2}\right]$$

$$\Rightarrow I = \frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I = \frac{\pi}{2}\left[\log \frac{1}{2}\right]$$

$$\Rightarrow I = \frac{\pi}{2} \log \frac{1}{2}$$

Example - 2.5 -

Solve $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx$

$\sin^2 x$ is an even function. Recall if we replace x with -x and then get the same value as the original function then it is even function. $\sin^2(-x) = \sin^2 x$

So we apply $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$$I = 2\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$
$$= 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx$$
$$= \int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$
$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{2}$$

We could have also done

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx \quad \text{- (1)}$$
$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right)dx$$
$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx \quad \text{- (2)}$$

And then as before

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right)dx$$
$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\cdot dx$$
$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$
$$\Rightarrow 2I = \frac{\pi}{2}$$
$$\Rightarrow I = \frac{\pi}{4}$$

So result is 2 X π/4 = π/2

But ideally I would have solved these problems by using gamma function show that

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

where m and n are integers.

Proof

Case I. When $n = 0$. Then

$$\int_0^{\pi/4} \sin^m x \cos^n x \, dx$$

$$= \int_0^{\pi/2} \sin^m x \, dx$$

$$= \left[-\frac{\sin^{m-1} x \cos x}{m}\right]_0^{\pi/2} + \frac{m-1}{m}\int_0^{\pi/2} \sin^{m-2} x \, dx$$

$$= \frac{m-1}{m}\int_0^{\pi/2} \sin^{m-2} x \, dx$$

Learn more about Gamma function at

https://zookeepersblog.wordpress.com/gamma-function-integral-calculus/

So $\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \cos^2 x \, dx = \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{2+0+2}{2}\right)}$ recall $\Gamma\tfrac{1}{2} = \sqrt{\pi}$

$\Gamma\, 3/2 = \tfrac{1}{2}\Gamma\,\tfrac{1}{2}$ because $\Gamma\,(n+1) = n\,\Gamma\, n$ 3/2 is (½ + 1) so n = ½

$\Gamma\, 2 = 1$ because $\Gamma\, 1 = 1$ So Integral = $\left(\left(\left(\tfrac{1}{2}\right)\sqrt{\pi}\right)\left(\sqrt{\pi}\right)\right)/\,2 = \pi/4$

Example - 2.6 - These type of problems are known as removal of x

Solve $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

$$\text{Let } I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} \quad \text{- (1)}$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)}\,dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x}\,dx \quad \text{- (2)}$$

Adding (1) and (2)

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin x}\,dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \left\{\sec^2 x - \tan x \sec x\right\} dx$$

$$\Rightarrow 2I = \pi \left[\tan x - \sec x\right]_0^{\pi}$$

$$\Rightarrow 2I = \pi [2]$$

$$\Rightarrow I = \pi$$

Example - 2.7 -

Solve $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$

Sin^7 x is an odd function. Because $\text{Sin}^7(-x) = -\text{Sin}^7 x$ So we use $\int_{-a}^{a} f(x)\,dx = 0$

So answer is 0

Example - 2.8 -

Solve $\int_0^{2\pi} \cos^5 x dx$

$$\text{Let } I = \int_0^{2\pi} \cos^5 x dx \qquad \ldots(1)$$

$$\cos^5(2\pi - x) = \cos^5 x$$

We have

$$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, \text{ if } f(2a-x) = f(x)$$
$$= 0 \text{ if } f(2a-x) = -f(x)$$

$$\therefore I = 2\int_0^{\pi} \cos^5 x dx$$

$$\Rightarrow I = 2(0) = 0 \qquad \left[\cos^5(\pi - x) = -\cos^5 x\right]$$

Example - 2.9 -

Solve $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \quad \text{- (1)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx \quad \text{- (2)}$$

Adding (1) and (2)

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{1 + \sin x \cos x} dx$$

$$\Rightarrow I = 0$$

Example - 2.10 -

Solve $\int_0^{\pi} \log(1 + \cos x) dx$

$$\text{Let } I = \int_0^{\pi} \log(1 + \cos x) dx \quad \text{-(1)}$$

$$\Rightarrow \; I = \int_0^{\pi} \log(1 + \cos(\pi - x)) dx$$

$$\Rightarrow I = \int_0^{\pi} \log(1 - \cos x) dx \quad \text{-(2)}$$

Adding (1) and (2)

$$2I = \int_0^{\pi} \left\{ \log\left(1+\cos x\right) + \log\left(1-\cos x\right) \right\} dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log\left(1-\cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x \, dx$$

$$\Rightarrow 2I = 2\int_0^{\pi} \log \sin x \, dx$$

$$\Rightarrow I = \int_0^{\pi} \log \sin x \, dx \quad \text{- (3)}$$

$$\sin(\pi - x) = \sin x$$

$$\therefore I = 2\int_0^{\frac{\pi}{2}} \log \sin x \, dx \quad \text{- (4)}$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = 2\int_0^{\frac{\pi}{2}} \log \cos x \, dx \quad \text{- (5)}$$

Adding (4) and (5) we get

$$2I = 2\int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x + \log 2 - \log 2\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log 2\sin x \cos x - \log 2\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

Let 2x=t so 2dx = dt when x=0 t is 0 and when x = π/2 t = π

$$\therefore I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2}\log 2$$

$$\Rightarrow I = \frac{1}{2} I - \frac{\pi}{2}\log 2$$

$$\Rightarrow \frac{I}{2} = -\frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\pi \log 2$$

Example - 2.11 -

Solve $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$

Add with $I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx$

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

$$\Rightarrow 2I = \int_0^a 1\,dx$$

$$\Rightarrow 2I = [x]_0^a$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

Similarly

$I = \int_3^5 \frac{\sqrt{x}}{\sqrt{8-x}+\sqrt{x}}\ dx$ then I equals

(a) 1 (b) 2
(c) 3 (d) 3.5

Ans. (a)

Solution Using the property

$$\int_a^b f(x)\ dx = \int_a^b f(a+b-x)dx$$

we can write

$$I = \int_3^5 \frac{\sqrt{8-x}}{\sqrt{x}+\sqrt{8-x}}\ dx$$

Adding

$$2I = \int_3^5 \frac{\sqrt{x}+\sqrt{8-x}}{\sqrt{x}+\sqrt{8-x}}\ dx = \int_3^5 dx = x]_3^5$$

$$\Rightarrow \quad 2I = 5 - 3 = 2 \Rightarrow I = 1.$$

AIEEE (now known as IIT-JEE main) - 2002

$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$ is

(a) $\pi^2/4$ (b) π^2 (c) 0 (d) $\pi/2$

(b) : $2\int_{-\pi}^{\pi} \frac{x}{1+\cos^2 x} dx + 2\int_{-\pi}^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$

$= 0 + 2\int_{-\pi}^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$

$= 2 \cdot 2\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}$

$= 4\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$

$= 4 \times \frac{\pi}{2}\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$\left(\text{by using } \int_{0}^{\pi} xf(\sin x)\, dx = \frac{\pi}{2}\int_{0}^{\pi} f(\sin x)\, dx\right)$

$= 4\frac{\pi}{2} \times 2 \times \int_{0}^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$

$= 4\pi\ (\tan^{-1} \cos x)_{\frac{\pi}{2}}^{0}$ (By putting $\cos x = t$)

$= 4\pi \times \left(\frac{\pi}{4} - 0\right)$

$= \pi^2$

AIEEE (now known as IIT-JEE main) - 2005

The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx,\ a > 0$, is

(a) $\pi/2$ (b) $a\pi$ (c) 2π (d) π/a

Solution

(a) : Let $f(x) = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx \quad (a > 0) \quad ...(1)$

$\therefore\ f(x) = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}} dx$

$\therefore\ \int_a^b f(x)dx = \int_a^b f(a+b-x)$

$\therefore\ f(x) = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \quad ...\ (2)$

$2f(x) = \int_{-\pi}^{\pi} \cos^2 x\, dx = 2\int_0^{\pi} \cos^2 x\, dx$

$= 2\times 2\int_0^{\pi/2} \cos^2 x\ dx,\ 2f(x) = 4 \times \frac{1}{2} \times \frac{\pi}{2}$

$\left[\text{By using } \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}.....\frac{1}{2}\times\frac{\pi}{2} \text{ if } n \text{ is even}\right]$

$f(x) = \frac{\pi}{2}$

Spoon feed with Sin^2 x

If $I = \int_{-\pi}^{\pi} \frac{\sin^2 x}{1+a^x} dx, \quad (1)$

$a > 0$, then I equals

(a) π (b) $\pi/2$
(c) $a\pi$ (d) $a\pi/2$

Ans. (b)

Solution As in Example 2,

$$I = \int_{-\pi}^{\pi} \frac{(\sin(-x))^2}{1+a^{-x}} dx$$

$$= \int_{-\pi}^{\pi} \frac{a^x \sin^2 x}{1+a^x} dx \quad (2)$$

Adding (1) and (2)

$$2I = \int_{-\pi}^{\pi} \sin^2 x\, dx$$

$$= 2\int_0^{\pi} \sin^2 x\, dx$$

$$= \int_0^{\pi} (1 - \cos 2x)\, dx$$

$$= \left(x - \frac{\sin 2x}{2}\right)_0^{\pi} = \pi$$

$$\Rightarrow I = \pi/2.$$

Walli's Formula

If n is a +ve integer then $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$ has the value

$$\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} \text{ if } n \text{ is odd}$$

and the value

$$\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{3}{4}\frac{1}{2} \cdot \frac{\pi}{2} \text{ if } n \text{ is even}$$

Proof

Let $I_n = \int_0^{\pi/2} \sin^n x\, dx$

$$= \int_0^{\pi/2} \sin^n \left(\frac{\pi}{2} - x\right) dx \quad \Bigg| \because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$$= \int_0^{\pi/2} \cos^n x\, dx$$

$$= \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$$

$$\Rightarrow \quad I_n = \frac{n-1}{n} I_{n-2} \quad \ldots (1)$$

Replacing n by $n - 2$, we get

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4} \quad \ldots (2)$$

Putting the value of I_{n-2} from (2) in (1), we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4} \quad \ldots (3)$$

Replace n by $n - 4$ in (1), we get

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6} \qquad \text{... (4)}$$

Putting the value of I_{n-4} from (4) in (3), we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6}$$

Proceeding in this manner, we see that two cases arise :

Case I. When n is odd, then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} \cdot I_1$$

Now $\quad I_1 = \int_0^{\pi/2} \sin x\, dx$

$$= (-\cos x)_0^{\pi/2}$$

$$= 1$$

$\therefore \quad I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3}$

Case II. When n is even, then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2} \cdot I_0$$

Now $\quad I_0 = \int_0^{\pi/2} \sin^0 x\, dx$

$$= (x)_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

$\therefore \quad I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2} \cdot \frac{\pi}{2}$

AIEEE (now known as IIT-JEE main) - 2006

The value of the integral, $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}\,dx$ is

(a) 1/2 (b) 3/2 (c) 2 (d) 1

Solution - (b)

$$\int_a^b \frac{f(x)}{f(a+b+x)+f(x)} = \int_a^b f(x)dx = \frac{b-a}{2}$$

$$\int_3^6 \frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}dx = \frac{6-3}{2} = \frac{3}{2}$$

AIEEE (now known as IIT-JEE main) - 2006

$\int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3+\cos^2(x+3\pi)]dx$ is equal to

(a) $\frac{\pi^4}{32}$ (b) $\frac{\pi^4}{32}+\frac{\pi}{2}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{2}-1$

Solution :

(c) : Let $I = \int_{\frac{-3\pi}{2}}^{\frac{-\pi}{2}} \left[(x+\pi)^3+\cos^2(x+3\pi)\right]dx$

Putting $x + \pi = z$

also $x = \frac{-\pi}{2} \Rightarrow z = \frac{\pi}{2}$ and $x = \frac{-3\pi}{2} \Rightarrow z = \frac{-\pi}{2}$

$\therefore$ $dx = dz$

and $x + 3\pi = z + 2\pi$

$\therefore$ $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} [z^3+\cos^2(2\pi+z)]dz$

$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} z^3 dz + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 z\,dz$

$= 0$ (an odd function) $+2\int_0^{\frac{\pi}{2}} \cos^2 z\,dz$

$= 0 + 2\frac{1}{2}\times\frac{\pi}{2}$

$\left\{\text{using fact } \int_0^{\frac{\pi}{2}} \sin^n x\,dx\right.$

$$= \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{1}{2}\times\frac{\pi}{2} & \text{if } n = 2m \\ \frac{n-1}{n}\frac{n-3}{n-2}\cdots\cdots\frac{2}{2} & \text{if } n = 2m+1 \end{cases}$$

$= \frac{\pi}{2}$

Type 3 - Special Definite Integral Formulae

Great mathematicians proved and Derived many interesting results. We have to know these results as of standard 12. Deriving all of these is not in course of IIT-JEE, or PU

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} \qquad a>0$$

$$\int_0^\infty x^n e^{-ax}\,dx = \begin{cases} \dfrac{\Gamma(n+1)}{a^{n+1}} & n>-1,\ a>0 \\ \dfrac{n!}{a^{n+1}} & a>0,\ n \text{ positive integer} \end{cases}$$

$$\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a} \qquad a>0$$

$$\int_0^\infty x e^{-x^2}\,dx = \frac{1}{2}$$

$$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}$$

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \qquad \int_0^\infty x^3 e^{-x^2}dx = \frac{1}{2}$$

$$\int_0^\infty x e^{-x^2}dx = \frac{1}{2} \qquad \int_0^\infty x^4 e^{-x^2}dx = \frac{3\sqrt{\pi}}{8}$$

$$\int_0^\infty x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{4} \qquad \int_0^\infty x^5 e^{-x^2}dx = 1$$

Or say to scare you more

$$\int_0^\pi \ln(a + b\cos x)\,dx = \pi \ln\left(\frac{a+\sqrt{a^2-b^2}}{2}\right)$$

$$\int_0^\pi \ln(a^2 - 2ab\cos x + b^2)\,dx = \begin{cases} 2\pi \ln a, & a \geqq b > 0 \\ 2\pi \ln b, & b \geqq a > 0 \end{cases}$$

$$\int_0^{\pi/4} \ln(1 + \tan x)\,dx = \frac{\pi}{8}\ln 2$$

$$\int_0^{\pi/2} \sec x \ln\left(\frac{1+b\cos x}{1+a\cos x}\right)dx = \tfrac{1}{2}\{(\cos^{-1}a)^2 - (\cos^{-1}b)^2\}$$

$$\int_0^a \ln\left(2\sin\frac{x}{2}\right)dx = -\left(\frac{\sin a}{1^2} + \frac{\sin 2a}{2^2} + \frac{\sin 3a}{3^2} + \cdots\right)$$

While the Indian toppers of IIT-JEE will know how to do these

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx \qquad I_n = \frac{n-1}{n} I_{n-2}$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx \qquad I_n = \frac{n-1}{n} I_{n-2}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin x}\,dx \qquad I_n = \frac{2(-1)^{n-1}}{2n-1} + I_{n-1}$$

$$\int_0^{\frac{\pi}{4}} \tan^n x\,dx \qquad I_n = \frac{1}{n-1} - I_{n-2}$$

$$\int_0^{\frac{\pi}{2}} e^{ax} \cos^n x\,dx \quad I_n = -\frac{a}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2} I_{n-2}$$

$$\int_0^{\frac{\pi}{2}} x^n \cos x\,dx \qquad I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

Some of the Derivations are given at

https://zookeepersblog.wordpress.com/iit-jee-integral-calculus-indefinite-definite-integration-skmclasses-south-bangalore-subhashish-sir/

Solve $\int_0^{\frac{\pi}{4}} \log\left(1+\tan x\right)dx$ (This was there in the formula list above)

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \log\left(1+\tan x\right)dx$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan x}{1+\tan}\right\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\frac{2}{(1+\tan x)}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2\,dx - \int_0^{\frac{\pi}{4}} \log\left(1+\tan x\right)dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2dx - I$$

$$\Rightarrow 2I = \left[x\log 2\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow I = \frac{\pi}{8}\log 2$$

Type - 4 - Integration of a modulus function. To be done piece wise due to break or reversal of value(s) somewhere.

Example - 4.1 - Solve $\int_{-5}^{5} |x+2|\, dx$

Around x = -2 the value of (x + 2) flips. Student can solve x + 2 = 0 to get x = -2

In some cases there will be a Quadratic function inside the modulus. In those cases there may be two separate values around which the value of the expression flips from positive to negative, or vice versa. These are the real roots of the Quadratic Expression. If the roots of the Quadratic expression are imaginary then the expression is either positive or negative for all values of x

So | x + 2 | = x + 2 for all x > -2 or rather right side of -2 (Better written as -2 < x, as per number line)

And for x < -2 | x + 2 | = - (x + 2) = -x - 2 This ensures that | x + 2 | is always positive

Thus the integral has to be split from -5 till -2_ [meaning -5 till less than -2 or -2-δ where δ is very small positive number that tends to 0 (zero). Mathematically we write Lt δ -> 0]

While the other part will be -2+ to 5 [meaning -2+δ till 5 where δ is very small positive number that tends to 0 (zero).

So we have the solution as

$$\therefore I = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5}(x+2)dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$I = -\left[\frac{x^2}{2}+2x\right]_{-5}^{-2} + \left[\frac{x^2}{2}+2x\right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2}+2(-2)-\frac{(-5)^2}{2}-2(-5)\right]+\left[\frac{(5)^2}{2}+2(5)-\frac{(-2)^2}{2}-2(-2)\right]$$

$$= -\left[2-4-\frac{25}{2}+10\right]+\left[\frac{25}{2}+10-2+4\right]$$

$$= -2+4+\frac{25}{2}-10+\frac{25}{2}+10-2+4$$

$$= 29$$

Example - 4.2 - Try another one where modulus flips around 5

Solve $\int_{2}^{8} |x-5|\, dx$

X - 5 ≤ 0 in [2,5] and x - 5 ≥ 0 in [5,8], thus

$$I = \int_2^5 -(x-5)\,dx + \int_2^8 (x-5)\,dx$$
$$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$
$$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$
$$= 9$$

Spoon feed

If $I = \int_{-3}^{2} (|x+1| + |x+2| + |x-1|)\, dx$, then

I equals

(a) $\frac{31}{2}$ (b) $\frac{35}{2}$

(c) $\frac{37}{2}$ (d) $\frac{39}{2}$

Ans. (a)

Solution We can write

$$I = I_1 + I_2 + I_3$$

where $I_1 = \int_{-3}^{2} |x+1|\, dx$ etc.

Put $x + 1 = t$, so that

$$I_1 = \int_{-2}^{3} |t|\, dt = \int_{-2}^{0} (-t)\, dt + \int_0^3 t\, dt$$
$$= -\frac{1}{2}t^2\Big]_{-2}^{0} + \frac{1}{2}t^2\Big]_0^3 = \frac{13}{2}$$

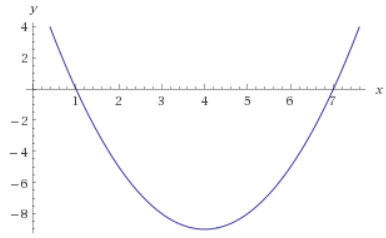
Similarly, $I_2 = I_3 = \frac{9}{2}$

Thus, $I = \frac{31}{2}$.

Example - 4.3 - Try to integrate modulus of Quadratic function

Let us cook the Quadratic Q(x) such that it has roots 1 and 7

So Q(x) will be (x - 1)(x - 7) = x^2 - 8x + 7

The graph will be 

It is obvious that Q(x) is +ve when x is less that 1 or when x is greater that 7

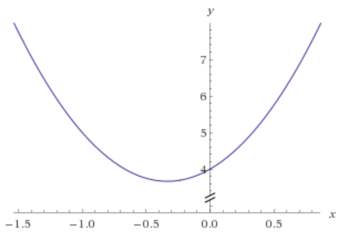
Q(x) is negative when x is in between 1 or 7 (1 < x < 7)

Now if we need to find $\int_{-10}^{11} |Q(x)| dx$ then we have to split from -10 to 1 then 1 to 7 and 7 to 11

So $\int_{-10}^{1} (x^2 - 8x + 7)\, dx + \int_{1}^{7} -(x^2 - 8x + 7)\, dx + \int_{7}^{11} (x^2 - 8x + 7)\, dx$

If the Quadratic function has imaginary roots $b^2 < 4ac$ (say b = 2 a = 3 and c = 4)

It will be above x axis always (a being positive)

Q(x) = $3x^2 + 2x + 4$ which will have a graph of 

So if we have to integrate from any lower limit to any higher limit of | $3x^2 + 2x + 4$ | if will be straight away done by integrating $3x^2 + 2x + 4$

AIEEE (now known as IIT-JEE main) - 2002

$\int_{\pi}^{10\pi} |\sin x| dx$ is

(a) 20 (b) 8 (c) 10 (d) 18

(d) : $\int_{\pi}^{10\pi} |\sin x| \, dx$

$= \int_{0}^{10\pi} |\sin x| \, dx - \int_{0}^{\pi} |\sin x| \, dx$

$= 10 \times 2 - 1 \times 2$

$= 18$ (Using period of $|\sin x| = \pi$)

AIEEE (now known as IIT-JEE main) - 2004

The value of $\int_{-2}^{3} |1-x^2| \, dx$ is

(a) 7/3 (b) 14/3 (c) 28/3 (d) 1/3

The value of the Quadratic flips around -1 and 1

(c) : $\int_{-2}^{3} \left|1 - x^2\right| dx = \int_{-2}^{3} \left|(1-x)(1+x)\right| dx$

Putting $1 - x^2 = 0 \therefore x = \pm 1$

Points $-2, -1, 1, 3$

$\therefore |1 - x^2| = \begin{cases} 1-x^2 & \text{if } |x|<1 \\ (1-(1-x^2)) & \text{if } x < -1 \text{ and } x \geq 1 \end{cases}$

$\therefore \int_{-2}^{3} \left|(1-x^2)\right| dx$

$= \int_{-2}^{-1} (x^2 - 1)dx + \int_{-1}^{1} (1 - x^2)dx + \int_{1}^{3} (x^2 - 1)dx$

$= \frac{4}{3} + 2\left(\frac{2}{3}\right) + \frac{20}{3} = \frac{28}{3}$

Example - 4.4 -

If $I = \int_{-\pi/6}^{\pi/6} \frac{\pi + 4x^5}{1 - \sin\left(|x| + \pi/6\right)} dx$, then I equals

(a) 4π (b) $2\pi + 1/\sqrt{3}$

(c) $2\pi - \sqrt{3}$ (d) $4\pi + \sqrt{3} - 1/\sqrt{3}$

Ans. (a)

Solution As $\frac{4x^5}{1 - \sin\left(|x| + \pi/6\right)}$ is an odd function, and

$\frac{\pi}{1 - \sin\left(|x| + \pi/6\right)}$ is an even function, we get

$$I = 2\pi \int_0^{\pi/6} \frac{dx}{1 - \sin(x + \pi/6)}$$

Put $x + \pi/6 = t$, $dx = dt$

$$I = 2\pi \int_{\pi/6}^{\pi/3} \frac{dt}{1 - \sin t} = 2\pi \int_{\pi/6}^{\pi/3} \frac{1 - \sin t}{\cos^2 t} dt$$

$$= 2\pi (\tan t + \sec t)]_{\pi/6}^{\pi/3}$$

$$= 2\pi \left\{\left(\sqrt{3} + 2\right) - \left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)\right\} = 4\pi$$

Type 5 - Cousins of ß functions

Beta functions are not directly in course. But in past 50 years, twice in IIT-JEE we had similar problems.

Let us start with an easy example - 5.1 - Which can be solved by $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Find $\int_0^1 x(1-x)^n\,dx$

$$\text{Let } I = \int_0^1 x(1-x)^n\,dx$$
$$\therefore I = \int_0^1 (1-x)(1-(1-x))^n\,dx$$
$$= \int_0^1 (1-x)(x)^n\,dx$$
$$= \int_0^1 \left(x^n - x^{n+1}\right)dx$$
$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

- >

$$= \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$
$$= \frac{(n+2)-(n+1)}{(n+1)(n+2)}$$
$$= \frac{1}{(n+1)(n+2)}$$

The same problem was asked in AIEEE (now known as IIT-JEE main) - 2003

So solving in another way for practice

The value of the integral $I = \int_0^1 x(1-x)^n\,dx$ is

(a) $\frac{1}{n+2}$ (b) $\frac{1}{n+1} - \frac{1}{n+2}$ (c) $\frac{1}{n+1} + \frac{1}{n+2}$ (d) $\frac{1}{n+1}$

(b) : $\int_0^1 x(1-x)^n \, dx$

Putting $x = \sin^2\theta$

$dx = 2 \sin\theta \cos\theta \, d\theta$

and $x = 0, \theta = 0$

$x = 1, \theta = \pi/2$

$$\therefore \int_0^1 x(1-x)^n dx = \int_0^{\pi/2} \sin^2\theta \cos^{2n}\theta \, (2\sin\theta\cos\theta) d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^{2n+1}\theta \, d\theta$$

$$\left[\text{Using} \int_0^{\pi/2} \sin^{2n+1}\theta \cos^{2n+1}\theta \, d\theta = \frac{[(2n)(2n-2)...2][(2n)(2n-2)...2]}{(4n+2)(4n)(4n-2)....2}\right]$$

$$\therefore 2\int_0^{\pi/2} \sin^3\theta \cos^{2n+1}\theta \, d\theta$$

$$= \frac{2[2\times(2n)(2n-2)(2n-4)...4.2]}{(2n+4)(2n+2)(2n)(2n-2)...4.2}$$

$$= \frac{2\times2\times1}{(2n+4)(2n+2)}$$

$$= \frac{1}{(n+2)(n+1)}$$

$$= \frac{1}{n+1} - \frac{1}{n+2} \quad \text{(by partial fraction)}$$

This was simplified version of Gamma Function. In fact Beta Function and Gamma Function are related.

Example - 5.2 -

Solve $\int_0^2 x\sqrt{2-x}\,dx$

$$\text{Let } I = \int_0^2 x\sqrt{2-x}\,dx$$

$$I = \int_0^2 (2-x)\sqrt{x}\,dx$$

$$= \int_0^2 \left\{2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right\}dx$$

$$= \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$$

$$= \frac{4}{3}(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}}$$

$$= \frac{4\times 2\sqrt{2}}{3} - \frac{2}{5}\times 4\sqrt{2}$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{40\sqrt{2} - 24\sqrt{2}}{15}$$

$$= \frac{16\sqrt{2}}{15}$$

Example - 5.3 -

Evalute $\int_0^{2a} x^{9/2}(2a-x)^{-1/2}\,dx.$

Solution :

$$I = \int_0^{2a} x^{9/2}(2a-x)^{-1/2}\,dx$$

Put $x = 2a\sin^2\theta$

$\therefore$ $dx = 4a\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2}(2a)^{9/2}\sin^9\theta\left(2a-2a\sin^2\theta\right)^{-1/2}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2}(2a)^{9/2}\cdot\sin^9\theta\cdot(2a)^{-1/2}\cos^{-1}\theta\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= (2a)^4\cdot 4a\cdot\int_0^{\pi/2}\sin^{10}\theta\,d\theta$$

$$= 64\,a^5\,\frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$ | Using Walli's formula

$$= \frac{63\,\pi\,a^5}{8}$$

Example - 5.4 -

If $I_n = \int_0^a (a^2-x^2)^n\,dx$, and $n > 0$, prove that $I_n = \frac{2na^2}{2n+1}I_{n-1}$.

Solution : We have

$$I_n = \int_0^a (a^2-x^2)^n\,dx$$

Put $x = a\sin\theta$

$\therefore$ $dx = a\cos\theta\,d\theta$

$$= \int_0^{\pi/2}\left(a^2-a^2\sin^2\theta\right)^n(a\cos\theta)\,d\theta$$

$$= a^{2n+1}\int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta$$

$$= a^{2n+1}\left[\left(\frac{\cos^{2n}\theta\sin\theta}{2n+1}\right)_0^{\pi/2} + \frac{2n}{2n+1}\int_0^{\pi/2}\cos^{2n-1}\theta\,d\theta\right]$$

$$= \frac{2n}{2n+1} a^{2n+1} \int_0^{\pi/2} \cos^{2n-1}\theta \, d\theta$$

$$= \frac{2na^2}{2n+1} \left\{ a^{2n-1} \int_0^{\pi/2} \cos^{2n-1}\theta \, d\theta \right\}$$

$$= \frac{2na^2}{2n+1} I_{n-1}$$

You can learn more at

https://zookeepersblog.wordpress.com/beta-function-integral-calculus-definite-indefinite-integration-skmclasses-south-bangalore-subhashish-sir/

Type 6 - Integration with greatest Integer functions (Also known as floor Function)

[2.3] = 2 while [2.9] is also 2 as it is the Integer equal or below (lesser) than the number

Note - most average students make an error in floor of negative number [-6.3] is -7 as -7 is the integer just lesser than -6.(whatever)

Floor function is also written as $\lfloor x \rfloor$

Example - 6.1 -

The value of the integral $\int_0^{2[x]} (x - [x])\, dx$ is

(a) $[x]$ (b) $\frac{1}{2}[x]$

(c) $3[x]$ (d) $2[x]$

Solution :

$$\int_0^{2[x]} (x-[x])\,dx = \int_0^{2[x]\cdot 1} (x-[x])\,dx$$

$$= 2[x]\int_0^1 (x-[x])\,dx$$

[$\because$ $x-[x]$ **is a periodic fn of period 1**]

$$= 2[x]\left(\left.\frac{x^2}{2}\right|_0^1 - \int_0^1 [x]\,dx\right) = 2[x]\left(\frac{1}{2}-0\right) = [x]$$

AIEEE (now known as IIT-JEE main) - 2002

$\int_0^{\sqrt{2}} [x^2]\,dx$ is

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$

(c) $\sqrt{2}-1$ (d) $\sqrt{2}-2$

(c) : $\int_0^{\sqrt{2}} [x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx$

$= 0 + \int_1^{\sqrt{2}} 1\,dx = \sqrt{2}-1$

Spoon feed

If $I = \int_0^{1.7} [x^2]\, dx$, then I equals

(a) $2{\cdot}4 + \sqrt{2}$ (b) $2 \cdot 4 - \sqrt{2}$

(c) $2{\cdot}4 - \sqrt{2}$ (d) $2 \cdot 4 - 1/\sqrt{2}$

Ans. (b)

Solution Put $x^2 = t$

or $x = \sqrt{t}$ or $dx = \dfrac{1}{2\sqrt{t}}\, dt$

$$\therefore \quad I = \int_0^{2.89} \frac{[t]}{2\sqrt{t}}\, dt$$

$$= \int_0^1 \frac{[t]}{2\sqrt{t}}\, dt + \int_1^2 \frac{[t]}{2\sqrt{t}}\, dt + \int_2^{2.89} \frac{[t]}{2\sqrt{t}}\, dt$$

$$= 0 + \int_1^2 \frac{1}{2\sqrt{t}}\, dt + \int_2^{2.89} \frac{2}{2\sqrt{t}}\, dt$$

$$= \sqrt{t}\,\Big]_1^2 + 2\sqrt{t}\,\Big]_2^{2.89}$$

$$= (\sqrt{2} - 1) + 2(1.7 - \sqrt{2})$$

$$= 2{\cdot}4 - \sqrt{2}$$

\

AIEEE (now known as IIT-JEE main) - 2002

$I_n = \int_0^{\pi/4} \tan^n x \, dx$, then $\lim_{n\to\infty} n[I_n + I_{n-2}]$ equals

(a) 1/2 (b) 1 (c) ∞ (d) 0

(b) : $I_n = \int_0^{\pi/4} \tan^n x \, dx$

$I_{n-2} = \int_0^{\pi/4} \tan^{n-2} x \, dx$

$\therefore I_n + I_{n-2} =$

$\int_0^{\pi/4} \tan^n x \, dx + \int_0^{\pi/4} \tan^{n-2} x \, dx$

$= \int_0^{\pi/4} \tan^{n-2} x \times (\sec^2 x - 1) \, dx + \int_0^{\pi/4} \tan^{n-2} x \, dx$

$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx$

$I_n + I_{n-2} = \dfrac{1}{n+1}$

$\therefore n(I_n + I_{n-2}) = \dfrac{1}{1 + 1/n}$

$\therefore \underset{n\to\infty}{\text{Lt}} \, n(I_n + I_{n-2}) = 1$

Example (Be Careful Just because [ ] is used do not assume greatest integer function. Solve the problem as greatest Integer only if it is told or as per context.)

The value of

$$I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + (x + 1) \cos (x + 1)]\, dx, \text{ is}$$

(a) -4 (b) -3

(c) -2 (d) -1

Ans. (c)

Solution We can write

$$I = \int_{-2}^{0} [(x + 1)^3 - 1 + (x + 1) \cos (x + 1)]dx$$

Put $x + 1 = t$, so that

$$I = \int_{-1}^{1} [t^3 - 1 + t \cos t]dt$$

$$= \int_{-1}^{1} (-1)dt = -t]_{-1}^{1} = -2$$

As $t^3 + t \text{ Cos } t$ is an odd function

AIEEE (now known as IIT-JEE main) - 2006

The value of $\int_1^a [x]\, f'(x)\, dx,\ a > 1,$ where $[x]$ denotes the greatest integer not exceeding x is

(a) $a f(a) - \{f(1) + f(2) + ... + f([a])\}$

(b) $[a] f(a) - \{f(1) + f(2) + ... + f([a])\}$

(c) $[a] f([a]) - \{f(1) + f(2) + ... + f(a)\}$

(d) $a f([a]) - \{f(1) + f(2) + ... + f(a)\}$

Solution :

(b) : $\int_2^a [x] f'(x)dx$, say $[a] = K$ such that $a > 1$

$= \int_1^2 1 f'(x)dx + \int_2^3 2 f'(x)dx + \ldots\ldots +$

$\int_{K-1}^{K} (K-1) f'(x)dx + \int_K^a K f'(x)dx$

$= f(2) - f(1) + 2[f(3) - f(2)] + 3[f(4) - f(3)] + \ldots$

$(K-1)\,[f(K) - f(K-1)] + K[f(a) - f(K)]$

$= -\,[f(1) + f(2) + \ldots\ldots + f(K)] + K f(a)$

$= [a]\, f(a) - [f(1) + f(2) + \ldots\ldots + f([a])]$

Example - 6.2 -

$$\text{If } I = \int_{-1}^{1} \left([x^2] + \log\left(\frac{2+x}{2-x}\right) \right) dx \qquad (1)$$

where $[x]$ denotes the greatest integer $\leq x$, then I equals

(a) -2 (b) -1

(c) 0 (d) 1

Ans. (c)

Solution As $\log\left(\frac{2+x}{2-x}\right)$ is an odd function, we can write

$$I = \int_{-1}^{1} [x^2]dx + 0$$

But for $-1 < x < 1$, $0 \leq x^2 < 1$ and thus, $[x^2] = 0$

$\therefore \quad I = 0.$

Example - 6.3 -

$\int_{-2}^{2}[x^2]\,dx$ is equal to

(a) $10-2\sqrt{3}-2\sqrt{2}$ (b) $10+2\sqrt{3}-2\sqrt{2}$

(c) $10-2\sqrt{3}+2\sqrt{2}$ (d) none of these

Solution

(a). $\int_{-2}^{2}[x^2]\,dx = 2\int_{0}^{2}[x^2]\,dx \quad [\because \text{ integrand is even}]$

$$=2\left[\int_0^1[x^2]\,dx+\int_1^{\sqrt{2}}[x^2]\,dx+\int_{\sqrt{2}}^{\sqrt{3}}[x^2]\,dx+\int_{\sqrt{3}}^{2}[x^2]\,dx\right]$$

$$\left[\because [x^2]=0 \text{ if } 0\le x<1; 1 \text{ if } 1\le x<\sqrt{2}; 2 \text{ if } \sqrt{2}\le x<\sqrt{3}; 3 \text{ if } \sqrt{3}\le x<2\right]$$

$$=2\left[\int_0^1 0\,dx+\int_1^{\sqrt{2}}1\,dx+\int_{\sqrt{2}}^{\sqrt{3}}2\,dx+\int_{\sqrt{3}}^{2}3\,dx\right]$$

$$=2\,(x)\Big|_1^{\sqrt{2}}+4\,(x)\Big|_{\sqrt{2}}^{\sqrt{3}}+6\,(x)\Big|_{\sqrt{3}}^{2}$$

$$=\left(\mathbf{10-2\sqrt{3}-2\sqrt{2}}\right).$$

Type - 7 - Problems with functions, derivatives, with some given conditions etc. These are more common to be asked in various Engineering entrance exams.

AIEEE (now known as IIT-JEE main) - 2003

Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ be a function that satisfies $f(x) + g(x) = x^2$. Then the value of the integral $\int_0^1 f(x)g(x)dx$ is

(a) $e + \frac{e^2}{2} - \frac{3}{2}$ (b) $e - \frac{e^2}{2} - \frac{3}{2}$

(c) $e + \frac{e^2}{2} + \frac{5}{2}$ (d) $e - \frac{e^2}{2} - \frac{5}{2}$

(b) : As $f(x) = f'(x)$ and $f(0) = 1$

$\Rightarrow \frac{f'(x)}{f(x)} = 1$

$\Rightarrow \log(f(x)) = x$

$\Rightarrow f(x) = e^x + k$

$\Rightarrow f(x) = e^x$ as $f(0) = 1$

Now $g(x) = x^2 - e^x$

$$\therefore \int_0^1 f(x)g(x)\,dx = \int_0^1 e^x(x^2 - e^x)\,dx$$

$$= \int_0^1 x^2 e^x dx - \int_0^1 e^{2x} dx$$

$$= [(x^2 - 2x + 2)e^x]_0^1 - \left(\frac{e^{2x}}{2}\right)_0^1$$

$$= (e - 2) - \left(\frac{e^2 - 1}{2}\right)$$

$$= e - \frac{e^2}{2} - \frac{3}{2}$$

Using $\int f^n(x)e^x\,dx = e^x[f^n(x) - f_1{}^n(x) + f_2{}^n(x) + \dots + (-1)^n f_n(x)]$

where $f_1, f_2, \dots f_n$ are derivatives of first, second ...n^{th} order.

Example - 7.1 -

Let $g(x) = \int_0^x f(t)\,dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$ and $0 \le f(t) \le \frac{1}{2}$ for $t \in [1, 2]$. Then,

(a) $-\frac{3}{2} \le g(2) \le \frac{1}{2}$ (b) $\frac{3}{2} \le g(2) \le \frac{5}{2}$

(c) $\frac{1}{2} \le g(2) \le \frac{3}{2}$ (d) none of these

Solution :

(c). We have, $g(2) = \int_0^2 f(t)\,dt$

$$= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \qquad \text{...(i)}$$

Now, $\frac{1}{2} \le f(t) \le 1$, for $t \in [0, 1]$

and, $0 \le f(t) \le \frac{1}{2}$, for $t \in [1, 2]$

$$\Rightarrow \quad \frac{1}{2}(1-0) \le \int_0^1 f(t)\,dt \le 1(1-0)$$

and, $$0(2-1) \le \int_1^2 f(t)\,dt \le \frac{1}{2}(2-1)$$

$$\left[\because \quad m \le f(x) \le M \text{ for } x \in [a, b] \right.$$

$$\left. \Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \right]$$

$$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)\,dt \le 1 \text{ and } 0 \le \int_1^2 f(t)\,dt \le \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \le \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{2} \le \int_0^2 f(t)\,dt \le \frac{3}{2} \quad \text{or} \quad \frac{1}{2} \le g(2) \le \frac{3}{2}$$

AIEEE (now known as IIT-JEE main) - 2003

If $f(y) = e^y$, $g(y) = y$; $y > 0$ and

$$F(t) = \int_0^t f(t-y)g(y)dy \text{, then}$$

(a) $F(t) = e^t - (1+t)$ (b) $F(t) = t\,e^t$

(c) $F(t) = te^{-t}$

(d) $F(t) = 1 - e^t(1+t)$.

(a) : From given $F(t) = \int_0^t f(t-y)\,g(y)\,dy$

$= \int_0^t e^{t-y} y\,dy$ (By replacing $y \to t - y$ in $f(y)$)

$F(t) = -\int_t^0 (t-\theta)e^\theta d\theta = \int_0^t (t-\theta)e^\theta d\theta$

$= (t\ e^\theta)_0^{\ t} - [(\theta - 1)\ e^\theta]_0^{\ t}$

$= t(e^t - 1) - (t - 1)e^t - 1$

$= e^t(t - t + 1) - t - 1$

$= e^t - (t + 1)$

Example - 7.2 -

Let $f(x)$ be a continuous function in $[-2, 2]$ such that

$f(x)+f(y)=f(x+y)$, then $\int_{-2}^{2} f(x)dx =$

(a) $2\int_{0}^{2} f(x)dx$ (b) 0

(c) $2f(2)$ (d) none of these

Solution :

(b). Since, $f(x)+f(y)=f(x+y)$ ...(1)

Replace y by $-x$

$\Rightarrow f(x)+f(-x)=f(x-x)$

$\Rightarrow f(x)+f(-x)=f(0)$ ...(2)

Also, using (1), we have

$f(0)+f(0)=f(0+0)=f(0)$

$\Rightarrow f(0)=0$

$\therefore f(-x)=-f(x)$ {using (2)}

$\Rightarrow \int_{-2}^{2} f(x)dx=0$

AIEEE (now known as IIT-JEE main) - 2003

If $f(a+b-x) = f(x)$, then $\int_a^b x f(x)dx$ is equal to

(a) $\frac{a+b}{2}\int_a^b f(x)dx$ (b) $\frac{b-a}{2}\int_a^b f(x)dx$

(c) $\frac{a+b}{2}\int_a^b f(a+b-x)dx$

(d) $\frac{a+b}{2}\int_a^b f(b-x)dx.$

(a), (c) : Let $I = \int_a^b x f(x)\, dx$

$$I = \int_a^b (a+b-x)\, f(a+b-x)\, dx$$

$$I = \int_a^b (a+b)\, f(a+b-x)\, dx - \int_a^b x f(a+b-x)\, dx$$

$$I = \int_a^b (a+b) f(x)dx - \int_a^b x f(x)dx$$

$$\therefore\ I = \frac{a+b}{2}\int_a^b f(x)\,dx = \frac{a+b}{2}\int_a^b f(a+b-x)dx$$

AIEEE (now known as IIT-JEE main) - 2003

Let $\frac{d}{dx}F(x) = \left(\frac{e^{\sin x}}{x}\right), x > 0.$

If $\int_1^4 \frac{3}{x}e^{\sin x^3} dx = F(k) - F(1)$, then one of the possible values of k is

(a) 16 (b) 63 (c) 64 (d) 15

$$\Rightarrow \int_1^4 \frac{3x^2}{x^3} e^{\sin x^3} dx = F(k) - F(1)$$

$$\Rightarrow \int_1^{64} \frac{e^{\sin z}}{z} dz = F(k) - F(1) \text{ where } (x^3 = z)$$

$$\Rightarrow [F(z)]_1^{64} = F(k) - F(1)$$

$$\Rightarrow F(64) - F(1) = F(k) - F(1)$$

$$\Rightarrow k = 64$$

AIEEE (now known as IIT-JEE main) - 2004

If $\int_0^\pi x\, f(\sin x)dx = A\int_0^{\pi/2} f(\sin x)dx$ then A is (a) $\pi/4$ (b) π (c) 0 (d) 2π

(b) : $\int_0^\pi x\, f(\sin x)dx = A\int_0^{\frac{\pi}{2}} f(\sin x)dx$

or $A\int_0^{\frac{\pi}{2}} f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx = \int_0^\pi x f(\sin x)dx$

$$\Rightarrow A\int_0^{\frac{\pi}{2}} f(\sin x)dx = \frac{\pi}{2}\times 2\int_0^{\frac{\pi}{2}} f(\sin x)dx$$

$$\Rightarrow A\int_0^{\frac{\pi}{2}} f(\sin x) = \pi\int_0^{\frac{\pi}{2}} f(\sin x)dx$$

$$\Rightarrow A = \pi$$

AIEEE (now known as IIT-JEE main) - 2004

If $f(x) = \dfrac{e^x}{1+e^x}$, $I_1 = \int\limits_{f(-a)}^{f(a)} xg\{x(1-x)\}\,dx$ and $I_2 = \int\limits_{f(-a)}^{f(a)} g\{x(1-x)\}\,dx$, then the value of $\dfrac{I_2}{I_1}$ is

(a) -1 (b) -3 (c) 2 (d) 1

Solution

(c) : As $f(x) = \dfrac{e^x}{1+e^x}$

$\therefore \quad f(a) = \dfrac{e^a}{1+e^a}$ and $f(-a) = \dfrac{e^{-a}}{1+e^{-a}}$

$\therefore \quad f(-a) + f(a) = 1$

Now $\int\limits_{f(-a)}^{f(a)} x\,g\{x(1-x)\}\,dx =$

$\int\limits_{f(-a)}^{f(a)} (1-x)g\{(1-x)(x)\}\,dx$

using $\int\limits_a^b f(x)dx = \int\limits_a^b f(a+b-x)dx$

$\Rightarrow \quad 2\int\limits_{f(-a)}^{f(a)} x\,g\{x(1-x)\}\,dx$

$= \int\limits_{f(-a)}^{f(a)} g\{(1-x)x\}\,dx$

$\Rightarrow \quad 2I_1 = I_2$

$\therefore \quad \dfrac{I_2}{I_1} = \dfrac{2}{1}$

AIEEE (now known as IIT-JEE main) - 2005

Let $F : R \to R$ be a differentiable function having $f(2) = 6$, $f'(2) = \left(\dfrac{1}{48}\right)$. Then $\lim\limits_{x\to 2} \int\limits_6^{f(x)} \dfrac{4t^3}{x-2}\,dt$ equals

(a) 36 (b) 24 (c) 18 (d) 12

Solution

(c) : $\lim\limits_{x\to 2} \int\limits_6^{f(x)} \dfrac{4t^3}{x-2}\,dt$ (0/0) form,

$= \lim\limits_{x\to 2} \dfrac{f'(x) \times 4(f(x))^3}{1}$

$= 4f'(2) \times (f(2))^3 = \dfrac{1}{48} \times 4 \times 6 \times 6 \times 6 = 18$

AIEEE (now known as IIT-JEE main) - 2006

$\int_0^{\pi} x f(\sin x)dx$ is equal to

(a) $\pi\int_0^{\pi} f(\cos x)dx$ (b) $\pi\int_0^{\pi} f(\sin x)dx$

(c) $\frac{\pi}{2}\int_0^{\pi/2} f(\sin x)dx$ (d) $\pi\int_0^{\pi/2} f(\cos x)dx$

Solution :

(d) : Let $I = \int_0^{\pi} xf(\sin x)dx$ ..... (i)

$I = \int_0^{\pi} (\pi - x) f(\sin x)dx$ ..... (ii)

using $\int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx$

By (i) & (ii) on adding

$\therefore\ I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx = 2\frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(\sin x)dx$

[using $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ if $f(2a-x) = f(x)$]

$= \pi\int_0^{\frac{\pi}{2}} f\left(\sin(\frac{\pi}{2}-x)\right)dx = \pi\int_0^{\frac{\pi}{2}} f(\cos x)dx$

AIEEE (now known as IIT-JEE main) - 2007

Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$, where $f(x) = \int_1^{x}\frac{\log t}{1+t}dt$

Then $F(e)$ equals

(a) 1 (b) 2 (c) 1/2 (d) 0

Solution :

(c) : $F(x) = \int_1^{x}\frac{\ln t}{1+t}dt + \int_1^{1/x}\frac{\ln t}{1+t}dt$

$F(x) = \int_1^{x}\left(\frac{\ln t}{1+t} + \frac{\ln t}{(1+t)t}\right)dt = \int_1^{x}\frac{\ln t}{t}dt = \frac{1}{2}(\ln x)^2$

$F(e) = 1/2.$

IIT - JEE 1998

If $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$, then the value of $f(1)$ is :

(A) $\frac{1}{2}$ (B) 0

(C) 1 (D) $-\frac{1}{2}$

Solution :

$$\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt,$$

Differentiating both sides w.r.t. x, we get

$$f(x)\cdot 1 = 1 - x\,f(x)\cdot 1$$

$\Rightarrow$ $(1+x)\,f(x) = 1$

$\Rightarrow$ $f(x) = 1/(1+x)$

$\Rightarrow$ $f(1) = \frac{1}{1+1} = \frac{1}{2}$

Thus (a) = ½ is the answer

Example - 7.3 -

If $f: R \to R$ is continuous and differentiable function such that

$$\int_{-1}^{x} f(t)\,dt + f'''(3)\int_{x}^{0} dt = \int_{1}^{x} t^3\,dt - f'(1)\int_{0}^{x} t^2\,dt + f''(2)\int_{x}^{3} t\,dt,$$

then the value of $f'(4)$ is

(a) $48 - 8f'(1) - f''(2)$
(b) $48 + 8f'(1) - f''(2)$
(c) $48 - 8f'(1) + f''(2)$
(d) $48 + 8f'(1) + f''(2)$

Solution

(a). From the given equation, we have

$$\int_{-1}^{x} f(t)\,dt - x f'''(3)$$

$$= \left(\frac{x^4}{4} - \frac{1}{4}\right) - f'(1)\frac{x^3}{3} + f''(2)\left(\frac{9}{2} - \frac{x^2}{2}\right)$$

Differentiating w.r.t. x, we get

$$f(x) - f'''(3) = x^3 - x^2 f'(1) - x f''(2)$$

Differentiating again, we have

$$f'(x) = 3x^2 - 2x f'(1) - f''(2)$$

$$\therefore \quad f'(4) = 48 - 8f'(1) - f''(2).$$

Example - 7.4 -

If f and g are two continuous functions being even and odd, respectively, then $\int_{-a}^{a} \frac{f(x)}{b^{g(x)} + 1}\,dx$ is equal to (a being any non-zero number and b is positive real number, $b \neq 1$)

(a) independent of f

(b) independent of g

(c) independent of both f and g

(d) none of these

Solution :

(b). Since, $\int_{-a}^{a} x f(x) dx = \int_0^a f(x) dx + \int_0^a f(-x) dx$

$$\therefore \int_{-a}^{a} \frac{f(x)}{b^{g(x)}+1} dx = \int_0^a \frac{f(x)}{b^{g(x)}+1} + \int_0^a \frac{f(-x)}{b^{g(-x)}+1}$$

$$= \int_0^a \frac{f(x)}{b^{g(x)}+1} dx + \int_0^a \frac{f(x)}{b^{-g(x)}+1}$$

$$= \int_0^a f(x) dx, \text{ which is independent of } g$$

Type - 8 - Differentiation of a Definite Integral often combined with L Hospital's rule.

Generally in most schools L Hospital's form itself is avoided. Differentiation of Definite Integrals with functions as lower and upper Limits are knows as Leibniz forms.

Learn more of Leibniz forms at

https://zookeepersblog.wordpress.com/leibnitz-rules-for-differentiation-of-integrals/

Leibniz Integral Rule

$$\frac{\partial}{\partial x}\left[\int_{y=a(x)}^{b(x)} f(x,y)\cdot dy\right] = \int_{y=a(x)}^{b(x)} \frac{\partial}{\partial x}\left[f(x,y)\right]\cdot dy + \left[f(x,y)\cdot\frac{\partial y}{\partial x}\right]_{y=a(x)}^{b(x)}$$

While the easier version is

$$\frac{d}{dx}\int_{f_1(x)}^{f_2(x)} g(x)dx = g(f_2(x))f_2'(x) - g(f_1(x))f_1'(x)$$

Most problems of Standard 12 (Engineering entrance) are doable by the 2nd (easier) version of Leibnitz.

IIT-JEE 2004

If f(x) is differentiable and given as $\int_0^{t^2} xf(x)dx = \frac{2}{5}t^5$ then find f(4/25)

Solution - Differentiate both sides with respect to t (using Leibnitz 2nd form)

$$t^2.f(t^2).2t = \frac{2}{5}.5t^4$$

Here if we put t = 2/5 we get t^2 = 4/25 So $f(t^2)$ = t Thus f(4/25) = 2/5

Example - 8.1 -

If $F(x) = \int_3^x \left(2 + \frac{d}{dt}\cos t\right) dt$ then $F'\left(\frac{\pi}{6}\right)$ is equal to

(a) 1/2 (b) 2 (c) 3/4 (d) 3/2

Ans. (d)

Solution $F(x) = \int_3^x (2 - \sin t)dt$ so $F'(x) = 2 - \sin x$.

Thus $F'(\pi/6) = 2 - 1/2 = 3/2$.

IIT-JEE 2007

Solve $\lim_{x\to\pi/4} \frac{\int_2^{\sec^2 x} f(t)dt}{x^2 - \pi^2/16}$

Solution : We can use L Hospital's rule because it is 0/0 form. Numerator and Denominator will be differentiated separately as per Leibnitz 2nd (simple) form

$$= \lim_{x\to\pi/4} \frac{f(\sec^2 x).(2\sec x).(\sec x.\tan x)}{2x} = \frac{8f(2)}{\pi}$$

AIEEE (now known as IIT-JEE main) - 2003

The value of $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \sec^2 t\,dt}{x\sin x}$ is

(a) 2 (b) 1 (c) 0 (d) 3

$$\textbf{(b)} : \underset{x\to 0}{\text{Lt}} \frac{(\tan t)_0^{x^2}}{x\sin x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\tan x^2}{x\sin x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\tan x^2}{x^2 \dfrac{\sin x}{x}}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\tan x^2}{x^2} \frac{1}{\underset{x\to 0}{\text{Lt}} \dfrac{\sin x}{x}} = 1 \times 1 = 1$$

Note in this problem Differentiation was avoided. The numerator was actually integrated and then the problem was solved. But often the function given cannot be integrated. In those cases Leibnitz Differentiation is an option.

A beautiful problem from **West Bengal JEE 2007**

Find $\lim\limits_{x\to\infty} \dfrac{\int_0^{2x} t e^{t^2} dt}{e^{4x^2}}$ West Bengal JEE 2007

(a) 0 (b) 2 (c) 1/2 (d) Infinity

Ans : (c)

Solution - We have

$$I = \lim_{x\to\infty} \frac{\int_0^{2x} t e^{t^2} dt}{e^{4x^2}} \quad \frac{\infty}{\infty} \text{ form}$$

$$I = \lim_{x\to\infty} \frac{2\times 2x\, e^{4x^2}}{e^{4x^2}\times 8x} \quad \text{using L Hospital's rule}$$

$= 1/2$

An alternate way of doing the above problem

$$I = \lim_{x \to \infty} \frac{\int_0^{2x} t e^{t^2} dt}{e^{4x^2}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\Rightarrow \quad I = \frac{1}{2} \lim_{x \to \infty} \frac{\int_0^{2x} e^{t^2} d(t^2)}{e^{4x^2}} = \frac{1}{2} \lim_{x \to \infty} \frac{\left[e^{t^2}\right]_0^{2x}}{e^{4x^2}}$$

$$\Rightarrow \quad I = \frac{1}{2} \lim_{x \to \infty} \frac{e^{4x^2} - 1}{e^{4x^2}} = \frac{1}{2} \lim_{x \to \infty} \left(1 - \frac{1}{e^{4x^2}}\right) = \frac{1}{2}(1 - 0) = \frac{1}{2}$$

Example Ratio of Integrals simplified individually

The value of $\lim_{m \to \infty} \frac{\int_0^{\pi/2} \sin^{2m} x \, dx}{\int_0^{\pi/2} \sin^{2m+1} x \, dx}$

(a) 0 (b) 1/2

(c) 2 (d) none of these

Ans. (d)

Solution We know that $I_{2n} = \int_0^{\pi/2} \sin^{2n} x \, dx$

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \ldots \times \frac{1}{2} \times \frac{\pi}{2},$$

$I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \ldots \times \frac{2}{3}$ and

Also, $I_{2m+1} = \frac{2m}{2m+1} I_{2m-1}$.

For all $x \in (0, \pi/2)$, $\sin^{2m-1} x > \sin^{2m} x > \sin^{2m+1} x$

Integrating from 0 to $\pi/2$, we get $I_{2m-1} \geq I_{2m} \geq I_{2m+1}$

whence $$\frac{I_{2m-1}}{I_{2m+1}} \geq \frac{I_{2m}}{I_{2m+1}} \geq 1 \quad \text{(i)}$$

Also $\frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}$. Hence $\lim_{m \to \infty} \frac{I_{2m-1}}{I_{2m+1}} = \lim_{m \to \infty} \frac{2m+1}{2m} = 1$.

From (i) and using sandwitch theorem we have $\lim_{m \to \infty} \frac{I_{2m}}{I_{2m+1}} = 1$.

Type - 9 - Some Summation problems which are solved by converting to Definite Integrals

AIEEE (now known as IIT-JEE main) - 2004

$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{r/n}$ is

(a) $1-e$ (b) $e-1$ (c) e (d) $e+1$ Recall the basics to solve these kinds of problems

Put 1/n as dx and r/n is substituted as x the limit r=1 to n changes to Integral 0 to 1

$$\textbf{(b)} : \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{r=n} \frac{1}{n} e^{\frac{r}{n}}$$

So $= \int_0^1 e^x dx = e - 1$

AIEEE (now known as IIT-JEE main) - 2005

$$\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+....+\frac{1}{n}\sec^2 1\right] \text{ equals}$$

(a) $\frac{1}{2}\operatorname{cosec}1$ (b) $\frac{1}{2}\sec 1$

(c) $\frac{1}{2}\tan 1$ (d) $\tan 1$.

Solution

(c) :

$$\lim_{n\to\infty}\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\left(\frac{4}{n^2}\right)+\ldots+\frac{1}{n}\sec^2 1$$

$$=\lim_{n\to\infty}\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\left(\frac{4}{n^2}\right)+\ldots+\frac{n}{n^2}\sec^2\left(\frac{n^2}{n^2}\right)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{r=n}\left(\frac{r}{n^2}\right)\sec^2\left(\frac{r}{n}\right)^2 = \lim_{n\to\infty}\sum_{r=0}^{r=n}\frac{1}{n}\left(\frac{r}{n}\right)\sec^2\left(\frac{r}{n}\right)^2$$

$$=\int_0^1 x\sec^2(x^2)dx = \frac{1}{2}\tan 1.$$

Type - 10 - Inequality of Definite Integrals

Schwarz-Bunyakovsky Inequality of Definite Integrals

If $f(x)$ and $g(x)$ are integrable on the interval (a, b), then $\int_a^b f(x)g(x)\,dx \le \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}$

For example

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx < \left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin x\,dx\right)^{\frac{1}{2}}$$

$$\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx < \sqrt{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\sin x\,dx\right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}$$

Example - 10.1 -

The value of the integral

$$\int_1^2 \sqrt{(2x+3)(3x^2+4)}\, dx \text{ cannot exceed}$$

(a) $\sqrt{48}$ (b) $\sqrt{66}$

(c) $\sqrt{73}$ (d) none of these

Solution

(b). $\int_1^2 \sqrt{(2x+3)(3x^2+4)}\, dx$

$$\leq \sqrt{\int_1^2 (2x+3)\, dx \cdot \int_1^2 (3x^2+4)\, dx}$$

$$\left[\because \left|\int_a^b f(x)\cdot g(x)\, dx\right| \leq \sqrt{\int_a^b f^2(x)\, dx \cdot \int_a^b g^2(x)\, dx}\right]$$

$$= \sqrt{[x^2+3x]_1^2 \cdot [x^3+4x]_1^2} = \sqrt{6\times 11} = \sqrt{66}$$

Example - 10.2 -

Show that $0 \leq \int_0^1 \frac{x\, dx}{x^3+16} \leq 1/17$

Solution :

$0 < x < 1$ means x varies between 0 to 1 where x is a fraction. So $x^3 < x^2$ Thus $x^3 + 1 < x^2 + 1$

⇨ $1/(x^3 + 1) > 1/(x^2 + 1)$

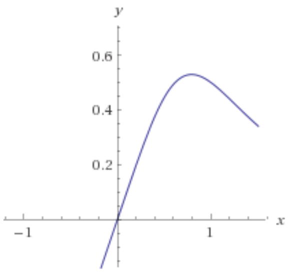
⇨ $\int_0^1 \frac{x\, dx}{x^2+16} < \int_0^1 \frac{x\, dx}{x^3+16}$

The function $f(x) = \frac{x}{x^3+16}$ is an increasing function on [0, 1] So min f(x) = f(0) = 0 and max f(x) = f(1) = 1/17

Referring to the property - If the function f (x) increases and has a concave graph in the interval [a, b], then

$$(b-a)\,f(a) < \int_a^b f(x)\,dx < (b-a)\,\frac{f(a)+f(b)}{2}$$

Or min (b - a) < Integral < Max (b - a)

Graph of y = $\frac{x}{x^3+16}$ is  (The scale of y axis is distorted)

Thus min (b - a) = 0 (1 - 0) = 0 and Max (b - a) = (1/17) (1 - 0) = 1/17 = 0.058823529

AIEEE (now known as IIT-JEE main) - 2005

If $I_1 = \int_0^1 2^{x^2}\,dx,\ I_2 = \int_0^1 2^{x^3}\,dx,\ I_3 = \int_1^2 2^{x^2}\,dx$ and $I_4 = \int_1^2 2^{x^3}\,dx$ then

(a) $I_1 > I_2$ (b) $I_2 > I_1$ (c) $I_3 > I_4$ (d) $I_3 = I_4$

Solution

(a) : For $0 < x < 1$, $x^2 > x^3$ $\therefore$ $\boxed{2^{x^2} > 2^{x^3}}$

and for $1 < x < 2$, $x^3 > x^2$ $\therefore$ $2^{x^3} > 2^{x^2}$

i.e. $\boxed{2^{x^2} < 2^{x^3}} \Rightarrow I_3 < I_4$

as $2^{x^2} > 2^{x^3}$

$\therefore \int_0^1 2^{x^2}\,dx > \int_0^1 2^{x^3}\,dx$

$\therefore\ I_1 > I_2.$

Example - 10.3 -

$\int_0^1 \frac{dx}{1+x^2+2x^5}$ lies between

(a) $\frac{1}{4}$ and 1 (b) $\frac{1}{4}$ and $\frac{1}{2}$

(c) $\frac{1}{2}$ and 1 (d) none of these

Solution :

In the interval [0, 1], $f(x)$ is strictly decreasing, therefore, we have,

$$f(1) \le f(x) \le f(0), \text{ i.e., } \frac{1}{4} \le f(x) \le 1$$

$$\therefore (1-0)\frac{1}{4} \le \int_0^1 f(x)\,dx \le (1-0)\,1$$

i.e., $$\frac{1}{4} \le \int_0^1 f(x)\,dx \le 1$$

Do it again

$\int_0^1 \frac{dx}{1+x^2+2x^5}$ lies between

(a) $\frac{\pi}{6\sqrt{3}}$ and $\frac{\pi}{4}$ (b) $\frac{\pi}{3\sqrt{3}}$ and $\frac{\pi}{2}$

(c) $\frac{\pi}{3\sqrt{3}}$ and $\frac{\pi}{4}$ (d) none of these

Solution :

(c). We have,

$$1 + x^2 + 2x^5 \geq 1 + x^2$$

and $1 + x^2 + 2x^5 \leq 1 + x^2 + 2x^5 = 1 + 3x^2$

$$\therefore \quad \frac{1}{1+3x^2} \leq \frac{1}{1+x^2+2x^5} \leq \frac{1}{1+x^2}$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{1+3x^2} \leq \int_0^1 \frac{dx}{1+x^2+2x^5} \leq \int_0^1 \frac{dx}{1+x^2}$$

$$\Rightarrow \left[\frac{\tan^{-1}\sqrt{3}x}{\sqrt{3}}\right]_0^1 \leq \int_0^1 \frac{dx}{1+x^2+2x^5} \leq [\tan^{-1}x]_0^1$$

$$\Rightarrow \quad \frac{\pi}{3\sqrt{3}} \leq \int_0^1 \frac{dx}{1+x^2+2x^5} \leq \frac{\pi}{4}$$

So we see as per the limits given we have to choose the approach

-

Inequality of Definite Integrals is explained and discussed at

https://archive.org/details/InequalitiesOfIntegralsUpperLimitAndLowerLimitsCanBeCookedPart1

-

Example - 10.4 -

$\int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}}$ belongs to the interval

(a) $\left[0, \frac{\pi}{6}\right]$ (b) $\left[\frac{\pi}{6}, \frac{\pi}{4\sqrt{2}}\right]$

(c) $\left[\frac{\pi}{4\sqrt{2}}, \frac{\pi}{2}\right]$ (d) none of these

Solution :

(b). Let $f(x) = \frac{1}{\sqrt{4-x^2-x^3}}$

Since $4-x^2 \geq 4-x^2-x^3 \geq 4-2x^2 > 1 \;\forall\; x \in [0,1]$

$\therefore \sqrt{4-x^2} \geq \sqrt{4-x^2-x^3} \geq \sqrt{4-2x^2} > 1 \;\forall\; x \in [0,1]$

$\Rightarrow \frac{1}{\sqrt{4-x^2}} \leq \frac{1}{\sqrt{4-x^2-x^3}} \leq \frac{1}{\sqrt{4-2x^2}} \;\forall\; x \in [0,1]$

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4-x^2}} \leq \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \int_0^1 \frac{dx}{\sqrt{2-2x^2}}$

$\Rightarrow \left|\sin^{-1}\frac{x}{2}\right|_0^1 \leq \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \frac{1}{\sqrt{2}}\left|\sin^{-1}\frac{x}{\sqrt{2}}\right|_0^1$

So $\frac{\pi}{6} \leq \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \frac{\pi}{4\sqrt{2}}$

AIEEE (now known as IIT-JEE main) - 2007

Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$.

Then which one of the following is true?

(a) $I > \frac{2}{3}$ and $J < 2$ (b) $I > \frac{2}{3}$ and $J > 2$

(c) $I < \frac{2}{3}$ and $J < 2$ (d) $I < \frac{2}{3}$ and $J > 2$

Solution :

(c) : In the interval of integration $\sin x < x$

$$I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx = \int_0^1 \sqrt{x}\, dx = \left[\frac{2}{3} x^{3/2}\right]_0^1 = \frac{2}{3}$$

$\therefore I < \frac{2}{3}$

Also $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_0^1 = 2$

$\therefore \quad J < 2$

Example - 10.5 -

If $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, then

(a) $\frac{1}{2} < I < \frac{1}{3}$ (b) $\frac{1}{4} < I < \frac{1}{3}$

(c) $\frac{1}{4} < I < 1$ (d) none of these

Solution :

(c). Let $f(x) = 2x^3 - 9x^2 + 12x + 4$, then $f(x)$ is a decreasing function on the interval $[1, 2]$.

$\therefore \quad 8 = f(2) < f(x) < f(1) = 9.$

$\therefore \quad \frac{1}{3} < \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}} < \frac{1}{\sqrt{8}}$

$\Rightarrow \quad \frac{1}{3}\int_1^2 dx < \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}} < \frac{1}{\sqrt{8}}\int_1^2 dx$

$\Rightarrow \quad \frac{1}{4} < \frac{1}{3} < I < \frac{1}{\sqrt{8}} < 1$

Hence, $\frac{1}{4} < I < 1.$

Example - 10.6 -

Let f be a real valued function satisfying $f(x) + f(x+6) = f(x+3) + f(x+9)$. Then, $\int_x^{x+12} f(t)\,dt$ is

(a) a linear function

(b) an exponential function

(c) a constant function

(d) none of these

Solution :

(c). Given $f(x) + f(x+6) = f(x+3) + f(x+9)$

Put $x = x + 3$, we get

$f(x + 3) + f(x + 9) = f(x + 6) + f(x + 12)$

$\Rightarrow \quad f(x) = f(x + 12)$

Let $\quad g(x) = \int_x^{x+12} f(t)\,dt \Rightarrow g'(x) = f(x+12) - f(x) = 0$

$\Rightarrow g(x)$ is a constant function.

Example - 10.7 -

If $f(x) = x + \int_0^1 (xy^2 - x^2 y)\, f(y)\, dy$, then $f(x)$ attains a minimum at

(a) $x = \frac{8}{9}$ (b) $x = -\frac{8}{9}$

(c) $\frac{9}{8}$ (d) $-\frac{9}{8}$

Solution :

(d). Given

$$f(x) = x + x\int_0^1 y^2 f(y)\,dy - x^2 \int_0^1 y f(y)\,dy$$

$$= x\left(1 + \int_0^1 y^2 f(y)\,dy\right) - x^2\left(\int_0^1 y f(y)\,dy\right)$$

$\Rightarrow$ $f(x)$ is a quadratic expression;

$\Rightarrow$ $f(x) = ax + bx^2$ or $f(y) = ay + by^2$ ...(1)

where, $a = 1 + \int_0^1 y^2 f(y)\,dy$

$$= 1 + \int_0^1 y^2 (ay + by^2)\,dy$$

$$= 1 + \left(\frac{ay^4}{4} + \frac{by^5}{5} \right)_0^1 = 1 + \left(\frac{a}{4} + \frac{b}{5} \right)$$

$\Rightarrow \quad 20a = 20 + 5a + 4b \quad$ or $\quad 15a - 4b = 20 \qquad$ ...(2)

and, $\quad b = \int_0^1 y\, f(y)\, dy = \int_0^1 y \cdot (ay + by^2)\, dy$

$$= \left(\frac{ay^3}{3} + \frac{by^4}{4} \right)_0^1 = \frac{a}{3} + \frac{b}{4}$$

$\Rightarrow \quad 12b = 4a + 3b \quad$ or $\quad 9b - 4a = 0 \qquad$ ...(3)

From (2) and (3),

$$a = \frac{180}{119},\ b = \frac{80}{119}$$

$\therefore$ Equation (1) reduces to

$$f(x) = \frac{80x^2 + 180x}{119}$$

$\therefore \quad f'(x) = \dfrac{160x + 180}{119} = 0 \Rightarrow x = \dfrac{-9}{8}$

and, $\quad f''(x) = \dfrac{160}{119} > 0 \Rightarrow f(x)$ attains minimum at $x = \dfrac{-9}{8}$

Type - 11 - Finding Area or Volume by applying Definite Integrals

This topic is covered in detail separately, in another e-Book

Putting only one example from AIEEE (now known as IIT-JEE main) - 2008

Area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is ?

(a) $\frac{4}{3}$ (b) $\frac{5}{3}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

Solution :

We have to draw a graph quickly to visualize the intersections and thus the region that is being considered.

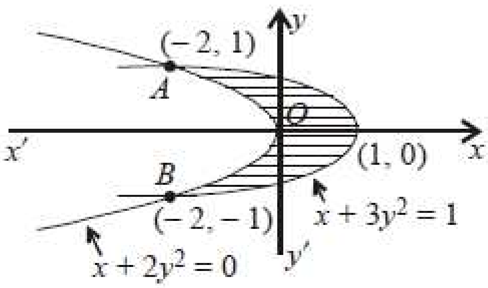
(a) : Solution $x + 2y^2 = 0$ and $x + 3y^2 = 1$ we have

$1 - 3y^2 = -2y^2 \Rightarrow y^2 = 1 \quad \therefore \quad y = \pm 1$

$y = -1 \Rightarrow x = -2$

$y = 1 \Rightarrow x = -2$

The bounded region is as under



$$\text{The desired area} = 2\int_0^1 [(1-3y^2)-(-2y^2)]dy$$

$$= 2\int_0^1 (1-y^2)dy = 2\left[y - \frac{y^3}{3}\right]_0^1$$

$$= 2 \times \frac{2}{3} = \frac{4}{3} \text{ sq. units}$$

Example - 11.1 -

The area bounded by the lines $y = 2, x = 1, x = a$ and the curve $y = f(x)$, which cuts the last two lines above the first line for all $a \geq 1$, is equal to $\frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right]$. Then, $f(x) =$

(a) $2\sqrt{2x}$ $x \geq 1$ (b) $\sqrt{2x}, x \geq 1$

(c) $2\sqrt{x}, x \geq 1$ (d) none of these

Solution :

(a). We are given

$$\int_1^a [f(x) - 2]\, dx = \frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right].$$

Differentiating w.r.t. a, we get

$$f(a) - 2 = \frac{2}{3}\left[\frac{3}{2}\sqrt{2a} \cdot 2 - 3\right]$$

$\Rightarrow$ $f(a) = 2\sqrt{2a}, a \geq 1$

$\therefore$ $f(x) = 2\sqrt{2x}, x \geq 1.$

This differentiation with respect to a or alpha is discussed below

Type - 12 - A reverse integration by Partial differentiation by assuming an unknown constant, to be variable. Often written as α

Example

The value of the integral $\int_0^1 \frac{x^b - 1}{\ln x}\, dx$ (b > 0) is

a) Ln | b | b) Ln | b + 1| c) 3 Ln | b | d) None of these

Answer (d)

Solution :

Let $I(b) = \int_0^1 \frac{x^b - 1}{\ln x}\, dx$ [Considering x as constant and partially differentiating with respect to b]

Recall d/dx of $a^x = a^x$ Ln a   So d/db of $x^b = x^b \ln x$

So $I'(b) = \int_0^1 \frac{x^b \ln x}{\ln x}\, dx = \int_0^1 x^b\, dx = \left.\frac{x^{b+1}}{b+1}\right|_0^1 = \frac{1}{b+1}$

Thus $I(b) = \int \frac{db}{b+1} = \ln | b + 1 | + c$ If b = 0 , then I(b) = 0 So c = 0

Hence I(b) = Ln | b + 1 |

Type - 13 - Problems with Fraction symbol { x }

{ 1.3 } = 0.3   { 9.1 } = 0.1   The fraction part of the number

Example - 13.1 -

The value of $\int_{-1}^{2} |[x] - \{x\}| dx$, where [x] is the greatest integer less then or equal to x and $\{x\}$ is the fractional part of x is

(a) 7/2   (b) 5/2

(c) 1/2   (d) 3/2

Ans. (a)

Solution For any $x \in$ R, x [x] + {x} so

[x] – {x} = 2 [x] – x. Thus

$$\int_{-1}^{2} |[x] - \{x\}| \, dx =$$

$$\int_{-1}^{0} |2[x] - \{x\}| \, dx + \int_{0}^{1} |2[x] - x| \, dx + \int_{1}^{2} |2[x] - x| \, dx$$

$$= \int_{-1}^{0} |2 + x| dx + \int_{0}^{1} |x| dx + \int_{1}^{2} |2 - x| \, dx$$

$$= \int_{-1}^{0} |2+x|\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx$$

$$= -\left(-2+\frac{1}{2}\right) + \frac{1}{2} + 2 - \frac{3}{2} = \frac{5}{2}$$

Example - 13.2 -

If $I_1 = \int_0^a [x]\,dx$ and $I_2 = \int_0^a \{x\}\,dx$, where $[x]$ and $\{x\}$ denote, respectively, the integral and fractional parts of x and a is a positive integer, then

(a) $I_2 = (a-1)\,I_1$ (b) $I_1 = (a-1)\,I_2$

(c) $I_1 = a\,I_2$ (d) $I_2 = a\,I_1$.

Solution :

(b). We have, $I_1 = \int_0^a [x]\,dx$

$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \cdots + \int_{a-1}^{a} (a-1)\,dx$$

$$= 1 + 2 + \cdots + (a-1) = \frac{a\,(a-1)}{2} \quad \ldots(1)$$

$$I_2 = \int_0^a \{x\}\,dx = \int_0^a (x-[x])\,dx = \int_0^a x\,dx - \int_0^a [x]\,dx$$

$$= \frac{x^2}{2}\bigg|_0^a - \frac{a\,(a-1)}{2} = \frac{a^2}{2} - \frac{a\,(a-1)}{2} = \frac{a}{2} \quad \ldots(2)$$

From (1) and (2), we have

$$\frac{I_1}{I_2} = (a-1) \cdot \quad \therefore I_1 = (a-1)\,I_2.$$

Example - 13.3 -

The value of $\int_0^1 (\{2x\}-1)(\{3x\}-1)\,dx$, where $\{\cdot\}$ denotes the fractional part is,

(a) $\frac{19}{72}$ (b) $\frac{31}{9}$

(c) $\frac{1}{8}$ (d) $\frac{72}{19}$

Solution :

(a). $\int_0^1 (\{2x\}-1)(\{3x\}-1)\,dx$

$$= \int_0^{1/3} (\{2x\}-1)(\{3x\}-1)\,dx + \int_{1/3}^{1/2} (\{2x\}-1)(\{3x-1\})\,dx$$

$$+ \int_{1/3}^{2/3} (\{2x\}-1)(\{3x\}-1)\,dx + \int_{2/3}^{1} (\{2x\}-1)(\{3x\}-1)\,dx$$

$$= \int_0^{1/3} (2x-1)(3x-1)\,dx + \int_{1/3}^{1/2} (2x-1)(3x-2)\,dx$$

$$+ \int_{1/2}^{2/3} (2x-2)(3x-2)\,dx + \int_{2/3}^{1} (2x-2)(3x-2)\,dx$$

$$= \int_0^{1/3} (6x^2-5x+1)\,dx + \int_{1/3}^{1/2} (6x^2-7x+2)\,dx$$

$$+ \int_{1/2}^{3/2} (6x^2-10x+4)\,dx + \int_{2/3}^{1} (6x^2-12x+6)\,dx$$

$$= \frac{19}{72}.$$

Example - 13.4 -

If $[x]$ and $\{x\}$ denote the integral and fractional parts of x, respectively, then $\int_0^x \left(x - [x] - \frac{1}{2} \right) dx$ is equal to

(a) $\frac{1}{2}\{x\}(\{x\}-1)$ (b) $\frac{1}{2}\{x\}(\{x\}+1)$

(c) $\{x\}(\{x\}-1)$ (d) none of these

Solution :

(a). We have,

$$\int_0^x \left(x - [x] - \frac{1}{2} \right) dx = \int_0^{[x]+\{x\}} \left(\{x\} - \frac{1}{2} \right) dx$$

$$= \int_0^{[x]} \left(\{x\} - \frac{1}{2} \right) dx + \int_{[x]}^{[x]+\{x\}} \left(\{x\} - \frac{1}{2} \right) dx$$

$$= [x] \int_0^1 \left(\{x\} - \frac{1}{2} \right) dx + \int_0^{\{x\}} \left(\{x\} - \frac{1}{2} \right) dx$$

$[\because \{x\}$ has period $1]$

$$= [x] \int_0^1 \left(x - \frac{1}{2} \right) dx + \int_0^{\{x\}} \left(x - \frac{1}{2} \right) dx$$

$$= [x] \left[\frac{x^2}{2} - \frac{x}{2} \right]_0^1 + \left[\frac{x^2}{2} - \frac{x}{2} \right]_0^{\{x\}}$$

$$= [x] \left(\frac{1}{2} - \frac{1}{2} \right) + \frac{\{x\}(\{x\}-1)}{2} = \frac{1}{2}\{x\}(\{x\}-1)$$

Type - 14 - Problems that don't fit into any standard form.

We need to solve rigorously and get the result, specific to the problem.

Such as

The value of the integral $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta$ is

(a) 0 (b) π

(c) 2π (d) cannot be determined

Solution :

Here we will use " i " as a tool to solve the problem. Euler Equation e^{ix} = Cos x + i Sin x helps us to modify the problem

(c). $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta$

$= \text{Real part of} \int_0^{2\pi} e^{\cos\theta} \{\cos(\sin\theta) + i \sin(\sin\theta)\}\, d\theta$

$= \text{Real part of} \int_0^{2\pi} e^{\cos\theta}\, e^{i\sin\theta}$

$= \text{Real part of} \int_0^{2\pi} e^{\cos\theta + i\sin\theta}\, d\theta$

$= \text{Real part of} \int_0^{2\pi} e^{e^{i\theta}}\, d\theta$

$$= \text{Real part of} \int_0^{2\pi} \left[1 + e^{i\theta} + \frac{e^{2i\theta}}{2!} + \frac{e^{3i\theta}}{3!} + \ldots\right] d\theta$$

$$= \text{Real part of} \int_0^{2\pi} \left[1 + (\cos\theta + i\sin\theta) + \frac{1}{2!}(\cos 2\theta + i\sin 2\theta) + \ldots\right] d\theta$$

$$= \int_0^{2\pi} \left[1 + \cos\theta + \frac{1}{2!}\cos 2\theta + \ldots\right] d\theta$$

$$= \left[\theta + \sin\theta + \frac{\sin 2\theta}{2.2!} + \ldots\right]_0^{2\pi} = 2\pi.$$

Example - 14.1 -

$$\text{If } I = \int_0^{\pi/2} \cos^n x \sin^n x \, dx = \lambda \int_0^{\pi/2} \sin^n x \, dx$$

then λ equals

(a) 2^{-n+1} (b) 2^{-n-1}

(c) 2^{-n} (d) 2^{-1}

Ans. (c)

Solution $I = \frac{1}{2^n} \int_0^{\pi/2} (2 \sin x \cos x)^n \, dx$

$$= \frac{1}{2^n} \int_0^{\pi/2} (\sin 2x)^n \, dx$$

Put $2x = \theta$, so that

$$I = \frac{1}{2^n} \int_0^{\pi} (\sin^n \theta) \frac{1}{2} \, d\theta$$

$$= \frac{1}{2^{n+1}} \left[\int_0^{\pi/2} [(\sin \theta)^n + (\sin (\pi - \theta))^n] \, d\theta\right.$$

using $\int_0^{2a} f(x) \, dx, = \int_0^{a} [f(x) + f(2a - x)] \, dx$

Sin (π - θ) = Sin θ so we can use gamma function for integrating $\text{Sin}^n \theta$

Practice example

If $\int_0^1 \frac{\sin t}{1+t}\, dt = \alpha$, then the value of the integral $\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi+2-t}\, dt$ in terms of α is given by

(a) 2α (b) -2α

(c) α (d) $-\alpha$

Solution :

(d). $\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi+2-t}\, dt = \frac{1}{2}\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{1+\left(2\pi-\frac{t}{2}\right)}\, dt$

$= 2\cdot\frac{1}{2}\int_0^1 \frac{\sin(2\pi-u)}{1+u}\, du$

$\left[\text{Putting } 2\pi-\frac{t}{2}=u \text{ so that } dt=-2\,du\right]$

$= -\int_0^1 \frac{\sin u}{1+u}\, du = -\int_0^1 \frac{\sin t}{1+t}\, dt = -\alpha$

An IIT-JEE problem from 70s

If $I_1 = \int_0^{\pi/2} \cos(\sin x)\, dx$; $I_2 = \int_0^{\pi/2} \sin(\cos x)\, dx$ and $I_3 = \int_0^{\pi/2} \cos x\, dx$, then

(a) $I_1 > I_3 > I_2$ (b) $I_3 > I_1 > I_2$

(c) $I_1 > I_2 > I_3$ (d) $I_3 > I_2 > I_1$

Solution :

(a). We have, $\sin x < x$ for $x > 0$

$\Rightarrow$ $\sin(\cos x) < \cos x$ for $0 < x < \pi/2$

$$\Rightarrow \int_0^{\pi/2} \sin(\cos x)\, dx < \int_0^{\pi/2} \cos x\, dx$$

$\therefore$ $I_3 > I_2$

Now, $\cos x < \cos \alpha$ if $x > \alpha$ and $x, \alpha \in \left[0, \frac{\pi}{2}\right]$

$\therefore$ $x > \sin x$

$\Rightarrow$ $\cos x < \cos(\sin x)$

$$\Rightarrow \int_0^{\pi/2} \cos x\, dx < \int_0^{\pi/2} \cos(\sin x)\, dx$$

$\therefore$ $I_3 < I_1$ ...(2)

$\therefore$ from (1) and (2) $\boldsymbol{I_1 > I_3 > I_2}$.

Example - 14.2 -

The natural number n (≤ 5) for which

$$I_n = \int_0^1 e^x (x-1)^n\, dx = 16 - 6e$$

is

(a) 2 (b) 3

(c) 4 (d) 5

Ans. (b)

Solution We have $I_0 = \int_0^1 e^x\, dx = e^x]_0^1 = e - 1$

and for $n \geq 1$,

$$I_n = e^x (x-1)^n]_0^1 - n \int_0^1 e^x (x-1)^{n-1}\, dx$$
$$= -(-1)^n - nI_{n-1}$$

$\therefore$ $I_1 = 1 - (1)I_0 = 1 - (e - 1) = 2 - e$

$I_2 = -1 - 2I_1 = -1 - 2(2 - e) = 2e - 5$

and $I_3 = 1 - 3I_2 = 1 - 3(2e - 5)$ = 16 - 6e So n = 3

Example - 14.3 -

If $b > a$ and $I = \int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$, then I equals

(a) $\pi/2$ (b) π
(c) $3\pi/2$ (d) 2π

Ans. (b)

Solution Put $t = \frac{1}{2}(x - a + x - b) = x - \frac{1}{2}(a + b)$, so that

$(x - a)(b - x) = (t + c)(c - t) = c^2 - t^2$

where $c = \frac{1}{2}(b - a)$.

Thus,

$$I = \int_{-c}^{c} \frac{dx}{\sqrt{(c^2 - t^2)}}$$

$$= 2\int_0^c \frac{dx}{\sqrt{(c^2 - t^2)}} = 2\sin^{-1}\left(\frac{t}{c}\right)\Bigg]_0^c$$

$$= 2[\sin^{-1}(1) - 0] = \pi$$

Example - 14.4 -

$$\text{If } b > a, \text{ and } I = \int_a^b \sqrt{\frac{x-a}{b-x}}\, dx,$$

then I equals

(a) $\frac{\pi}{2}(b-a)$ (b) $\pi(b-a)$

(c) $\pi/2$ (d) $2\pi(b-a)$

Ans. (a)

Solution Put $b - x = t^2$, so that

$$I = \int_{\sqrt{b-a}}^{0} \sqrt{\frac{b-t^2-a}{t^2}}\,(-2t)\,dt$$

$$= 2\int_0^c \sqrt{c^2-t^2}\,dt \text{ where } c = \sqrt{b-a}$$

$$= 2\left[\frac{1}{2}t\sqrt{c^2-t^2} + \frac{c^2}{2}\sin^{-1}\left(\frac{t}{c}\right)\right]_0^c$$

$$= 0 + c^2\sin^{-1}(1) - 0$$

$$= \frac{\pi}{2}(b-a).$$

Example 14.5 -

The mean value of the function $f(x) = \frac{1}{x^2+x}$ on the interval $[1, 3/2]$ is

(a) $\log(6/5)$ (b) $2\log(6/5)$

(c) 4 (d) $\log 3/5$

Ans. (b)

Solution Mean value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$$= \frac{1}{3/2-1}\int_1^{3/2}\frac{1}{x^2+x}\,dx = 2\int_1^{3/2}\left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$= 2(\log x - \log(x+1))\Big|_1^{3/2} = 2[\log(3/2) - \log(5/2) - (\log 1 - \log 2)]$$

$$= 2\log(6/5).$$

Example of Max function

The value of $\int_{-2}^{2} \max\{(1-x),(1+x),2\}\, dx$ is

(a) 8 (b) -8

(c) 9 (d) -9

Solution

(c). For $-2 \le x \le -1$, we have $1-x \ge 2$

and $1-x > 1+x$

$\therefore \quad \max\{(1-x),(1+x),2\} = 1-x.$

For $-1 < x < 1$, we have $0 < 1-x < 2$ and $0 < 1+x < 2$

$\therefore \quad \max\{(1-x),(1+x),2\} = 2.$

For $1 \le x \le 2$, we have $1+x \ge 2$ and $1+x > 1-x$

$\therefore \quad \max\{(1-x),(1+x),2\} = 1+x.$

$$\therefore \int_{-2}^{2} \max\{(1-x),(1+x),2\}\, dx$$

$$= \int_{-2}^{-1} (1-x)\, dx + \int_{-1}^{1} 2\, dx + \int_{1}^{2} (1+x)\, dx$$

$$= \left[x - \frac{x^2}{2}\right]_{-2}^{-1} + [2x]_{-1}^{1} + \left[x + \frac{x^2}{2}\right]_{1}^{2} = 9$$

Example - 14.6 -

If $\int_{0}^{100} f(x)\, dx = a$, then

$$\sum_{r=1}^{100} \left(\int_{0}^{1} f(r-1+x)\, dx \right)$$

(a) $100a$ (b) a

(c) 0 (d) $100a$

Solution :

(b). Let $I = \sum_{r=1}^{100} \left(\int_0^1 f(r-1+x)\,dx \right)$

$$\Rightarrow \quad I = \int_0^1 f(x)\,dx + \int_0^1 f(1+x)\,dx + \int_0^1 f(2+x)\,dx + \ldots + \int_0^1 f(99+x)\,dx$$

$$\Rightarrow \quad I = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx \ldots + \int_{99}^{100} f(x)\,dx$$

$$I = \int_0^{100} f(x)\,dx = a$$

Practice example

The value of $\int_1^{16} \tan^{-1} \sqrt{\sqrt{x} - 1}\; dx$ is

(a) $\frac{16\pi}{3} + 2\sqrt{3}$ (b) $\frac{4}{3}\pi - 2\sqrt{3}$

(c) $\frac{4}{3}\pi + 2\sqrt{3}$ (d) $\frac{16}{3}\pi - 2\sqrt{3}$

Ans. (d)

Solution Integrating by parts, the given integral is equal to

$$x \tan^{-1} \sqrt{\sqrt{x}-1}\Big|_1^{16} - \int_1^{16} \frac{x}{\sqrt{x}} \frac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}}\,dx$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_1^{16} \frac{dx}{\sqrt{\sqrt{x}-1}}$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_0^{\sqrt{3}} \frac{4t(1+t^2)}{t}\,dt \; (\sqrt{x} = 1 + t^2)$$

$$= \frac{16}{3}\pi - \left(\sqrt{3} + \sqrt{3}\right) = \frac{16}{3} - 2\sqrt{3}$$

Practice Example

For any $t \in \mathbb{R}$ and f a continuous function, let

$I_1 = \int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))\,dx$ and $I_2 = \int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\,dx$ then I_1/I_2 is equal to

(a) 2 (b) 1

(c) 4 (d) none of these

Ans. (b)

Solution $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} (2-x)\,f((2-x)(2-(2-x)))\,dx$

$$= \int_{\sin^2 t}^{1+\cos^2 t} (2-x)\,f(x(2-x))\,dx$$

$$=2\int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\,dx - \int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))\,dx = 2I_2 - I_1$$

Therefore, $2I_1 = 2I_2$ and so $I_1/I_2 = 1$.

Practice Example

If $\int_0^\infty e^{-ax}dx = \frac{1}{a}$, then $\int_0^\infty x^n e^{-ax}\,dx$ is

(a) $\frac{(-1)^n n!}{a^{n+1}}$ (b) $\frac{(-1)^n (n-1)!}{a^n}$

(c) $\frac{n!}{a^{n+1}}$ (d) none of these

Solution :

(c). Let $I_n = \int_0^\infty x^n e^{-ax}\,dx$

$$= \left[x^n \cdot \frac{e^{-ax}}{-a}\right]_0^\infty - \int_0^\infty nx^{n-1} \cdot \frac{e^{-ax}}{-a}\,dx$$

$$= -\frac{1}{a}\lim_{x\to\infty}\frac{x^n}{e^{ax}} + \frac{n}{a}I_{n-1}$$

$\therefore$ $I_n = \frac{n}{a}I_{n-1}$ $\left[\because \lim_{x\to\infty}\frac{x^n}{e^{ax}} = 0\right]$

$$= \frac{n}{a}\cdot\frac{n-1}{a}I_{n-2}$$

$$= \frac{n(n-1)(n-2)}{a^3}I_{n-3}$$

....

....

$$= \frac{n!}{a^n}\int_0^{\infty} e^{-ax}dx = \frac{n!}{a^{n+1}}$$

Practice Example

The value of $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$ is

(a) 0 (b) 2/3 (c) 4/3 (d) 1/3

Ans. (c)

Solution $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\, |\sin x|\, dx$

$= 2\int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\, |\sin x| dx$ (the integand is an even function)

$= 2\int_{0}^{\pi/2} \sqrt{\cos x} \sin x\, dx = -\frac{4}{3}(\cos x)^{3/2}\Big|_0^{\pi/2} = \frac{4}{3}.$

Practice Example

The value of $\int_1^a [x] f'(x) dx, a > 1$, where $[x]$ denotes the greatest integer not exceeding x is

(a) $af([a]) - \{f(1) + f(2) + ... + f(a)\}$

(b) $af(a) - \{f(1) + f(2) + ... + f([a])\}$

(c) $[a]f(a) - \{f(1) + f(2) + ... + f([a])\}$

(d) $[a]f([a]) - \{f(1) + f(2) + ... + f(a)\}$

Solution :

(c). $\int_1^a [x]f'(x)dx$

$= \int_1^2 f'(x)dx + 2\int_2^3 f'(x)dx + 3\int_3^4 f'(x)dx + \ldots + \int_{[a]-1}^{[a]} ([a]-1)f'(x)dx + [a]\int_{[a]}^{a} f'(x)dx$

$= (f(2)-f(1)) + 2(f(3)-f(2)) + 3(f(4)-f(3)) + \ldots + [a](f(a)-f([a]))$

$= [a]f(a) - \{f(1)+f(2)+f(3)+\ldots+f([a])\}$

Practice Example

The value of $\int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)\, d\theta$ is

(a) 1
(b) 0
(c) 2
(d) none of these

Ans. (b)

Solution $I = \int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)\, d\theta$

$= \int_{-\pi}^{3\pi} \log(\sec(2\pi-\theta) - \tan(2\pi-\theta))\, d\theta$

$= \int_{-\pi}^{3\pi} \log(\sec\theta + \tan\theta)\, d\theta.$

Thus $2I = \int_{-\pi}^{3\pi} [\log(\sec\theta - \tan\theta) + \log(\sec\theta + \tan\theta)]\, d\theta$

$= \int_{-\pi}^{3\pi} \log(\sec^2\theta - \tan^2\theta) d\theta = \int_{-\pi}^{3\pi} \log 1\, d\theta = 0.$

Practice Example

$$\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$$

Let $\quad I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$

Also, $\quad I = \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi - x)}}\,dx$

$$= \int_0^{2\pi} \frac{1}{1+e^{-\sin x}}\,dx$$

$$= \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}\,dx$$

Adding (*i*) and (*ii*), we have

$$2I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx + \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}\,dx$$

$$= \int_0^{2\pi} \frac{1+e^{\sin x}}{e^{\sin x}+1}\,dx = \int_0^{2\pi} 1\,dx$$

$$2I = |x|_0^{2\pi} = 2\pi$$

$$I = \pi$$

Solve a Simple Problem

$$\int \frac{3x+1}{2x^2+x+1}\,dx = \int \left(\frac{\frac{3}{4}(4x+1)+\frac{1}{4}}{2x^2+x+1} \right) dx$$

$$= \frac{3}{4}\int \left(\frac{4x+1}{2x^2+x+1} \right) dx + \frac{1}{8}\int \frac{dx}{\left(x^2+\frac{x}{2}+\frac{1}{2} \right)}$$

$$= \frac{3}{4}\log(2x^2+x+1) + \frac{1}{2\sqrt{7}}\tan^{-1}\frac{4x+1}{\sqrt{7}} + C$$

A routine problem asked in several exams

$$\int_0^{\sqrt{3}} \frac{1}{1+x^2}\cdot\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx =$$

(a) $\frac{7}{72}\pi^2$ (b) $\frac{3}{42}\pi^2$

(c) $\frac{17}{72}\pi^2$ (d) none of these

Solution :

(a). Let $I = \int_0^{\sqrt{3}} \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & \text{if } -1 \le x \le 1 \\ \pi - 2\tan^{-1}x, & \text{if } x > 1 \end{cases}$

$$\therefore\ I = \int_0^1 \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx + \int_1^{\sqrt{3}} \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

$$= \int_0^1 \frac{2\tan^{-1}x}{1+x^2}dx + \int_1^{\sqrt{3}} \frac{\pi - 2\tan^{-1}x}{1+x^2}dx$$

$$= 2\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx + \pi\int_1^{\sqrt{3}} \frac{1}{1+x^2}dx - 2\int_1^{\sqrt{3}} \frac{\tan^{-1}x}{1+x^2}dx$$

$$= 2\int_0^{\pi/4} t\,dt + \pi\,(\tan^{-1}x)_1^{\sqrt{3}} - 2\int_{\pi/4}^{\pi/3} t\,dt,$$

(Put $\tan^{-1}x = t$)

$$= 2\left\{\frac{t^2}{2}\right\}_0^{\pi/4} + \pi\,\{\tan^{-1}\sqrt{3} - \tan^{-1}1\} - 2\left\{\frac{t^2}{2}\right\}_{\pi/4}^{\pi/3}$$

$$= \frac{\pi^2}{16} + \pi\left\{\frac{\pi}{3} - \frac{\pi}{4}\right\} - \left\{\frac{\pi^2}{9} - \frac{\pi^2}{16}\right\} = \frac{7}{72}\pi^2.$$

Solve a problem

$$\int \frac{x}{(1-x)^{1/3} - (1-x)^{1/2}}\,dx \qquad \{\text{ The LCM of 2 and 3 is 6 }\}$$

Hence, substitute $1 - x = u^6$ Then, $dx = -6u^5\,du$

$$\Rightarrow I = \int \frac{1-u^6}{u^2 - u^3}(-6u^5\,du) = -6\int \frac{1-u^6}{1-u}u^3\,du$$

$$= -6\int (1 + u + u^2 + u^3 + u^4 + u^5)\,u^3\,du$$

$$= -6\left(\frac{1}{4}u^4 + \frac{1}{5}u^5 + \frac{1}{6}u^6 + \frac{1}{7}u^7 + \frac{1}{8}u^8 + \frac{1}{9}u^9\right) + c$$

Solve a Problem

The value of $\int_0^1 \frac{x}{x^2+16}\,dx$ lies in the interval $[a, b]$. The smallest such interval is

(a) $[0, 1]$ **(b)** $\left[0, \frac{1}{7}\right]$

(c) $\left[0, \frac{1}{17}\right]$ **(d)** none of these

Solution :

(c). Let $f(x) = \dfrac{x}{x^2+16}$

$\therefore \quad f'(x) = \dfrac{(x^2+16).1 - x.2x}{(x^2+16)^2}$

$= \dfrac{16-x^2}{(x^2+16)^2} \geq 0$

$\Rightarrow 16 \geq x^2 \Rightarrow x^2 \leq 16 \Rightarrow -4 \leq x \leq 4$

$\therefore$ $f(x)$ is monotonic increasing in $[-4, 4]$. Since $[0, 1]$, $\subseteq [-4, 4]$

$\therefore$ $f(x)$ is monotonic increasing in $[0, 1]$

$\therefore \quad M = \dfrac{1}{1+16} = \dfrac{1}{17}$ and $m = \dfrac{0}{0+16} = 0$

$$\therefore m\,(1-0) \leq \int_0^1 f(x)\,dx \leq M\,(1-0)$$

$$\Rightarrow 0\,(1-0) \leq \int_0^1 \frac{x}{x^2+16}\,dx \leq \frac{1}{17}(1-0)$$

$$\Rightarrow \quad 0 \leq \int_0^1 \frac{x\,dx}{x^2+16} \leq \frac{1}{17}$$

$\therefore$ The smallest such interval is $\left[0, \dfrac{1}{17}\right]$

Solve a Problem

Evaluate $\int \cos 2x \log(1+\tan x)dx$.

Solution:

Integrating by parts taking cos 2x as the 2nd function, the given integral

$$=\{\log(1+\tan x)\}\frac{\sin 2x}{2}-\int\frac{\sec^2 x}{1+\tan x}\cdot\frac{\sin 2x}{2}dx$$

$$=\frac{1}{2}\sin 2x\ \log(1+\tan x)-\int\frac{\sin x}{\sin x+\cos x}dx.$$

Now $\int\frac{\sin x\,dx}{\sin x+\cos x}$

$$=\frac{1}{2}\int\frac{(\sin x+\cos x)-(\cos x-\sin x)}{\sin x+\cos x}dx,$$

$$=\frac{1}{2}\int\left[1-\frac{\cos x-\sin x}{\sin x+\cos x}\right]dx=\frac{1}{2}[x-\log(\sin x+\cos x)].$$

Hence the given integral

$$=\frac{1}{2}\sin 2x\ \log(1+\tan x)-\frac{1}{2}[x-\log(\sin x+\cos x)].$$

Recall how to integrate Linear X root Quadratic in denominator

Find the value of the $\int\frac{dx}{(x+1)\sqrt{(1+2x-x^2)}}$

Putting $(x+1)=\frac{1}{t}$, so that $dx=-\frac{1}{t^2}\,dt$, $x=\frac{1-t}{t}$ and

$$(1+2x-x^2)=1+2\left(\frac{1-t}{t}\right)-\frac{(1-t)^2}{t^2}=\frac{2}{t^2}\left[\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2\right],$$

we get the value of the given integral transformed as

$$\int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\frac{2}{\sqrt{t}}\left[\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2\right]} = -\frac{1}{\sqrt{2}}\sin^{-1}\frac{t-1}{\left(\frac{1}{\sqrt{2}}\right)} + C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}\,x}{(x+1)} + C$$

Remember -

For the form $\int \frac{dx}{(Ax+B)^r\sqrt{(ax^2+bx+c)}}$ where r is a positive integer

we can substitute $Ax+B = \frac{1}{t}$

But for $\int \frac{dx}{(Ax+B)\sqrt{(ax+b)}}$ $\int \frac{dx}{(Ax^2+Bx+C)\sqrt{(ax+b)}}$ we have to substitute $ax + b = t^2$

So the Linear expression that in inside the root will be substituted

Another advanced example

Example Evaluate $\int \frac{dx}{x\sqrt{(1+x^n)}}$

Make the substitution $(1 + x^n) = t^2$ or $x^n = (t^2 - 1)$, so that $n\,x^{n-1}\,dx = 2t\,dt$, we get

$$\int \frac{2t\,dt}{n x^n t} = \frac{2}{n}\int \frac{dt}{(t^2-1)} = \frac{1}{n}\ln\left|\frac{t-1}{t+1}\right|$$

$$= \frac{1}{n}\ln\left|\frac{\sqrt{(1+x^n)}-1}{\sqrt{(1+x^n)}+1}\right| + C$$

Similarly

The value of integral $\int \frac{dx}{x\sqrt{1-x^3}}$ is given by

(a) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}+1}{\sqrt{1-x^3}-1}\right| + C$ (b) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^2}+1}\right| + C$

(c) $\frac{2}{3}\log\left|\frac{1}{\sqrt{1-x^3}}\right| + C$ (d) $\frac{1}{3}\log|1-x^3| + C$

Ans. (b)

Solution Put $1 - x^3 = t^2$. Then $-3x^2dx = 2t\,dt$ and the integral becomes

$$-\frac{1}{3}\int\frac{-3x^2dx}{x^3\sqrt{1-x^3}} = -\frac{1}{3}\int\frac{2t\,dt}{(1-t^2)t} = \frac{2}{3}\int\frac{dt}{t^2-1}$$

$$= \frac{2}{3}\left(\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|\right) + C = \frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + C$$

Solve a Problem

$\int\sqrt{\sec x - 1}\; dx$ is equal to

(a) $2\log\left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$

(b) $\log\left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$

(c) $-2\log\left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$

(d) none of these

(c). $\int \sqrt{\sec x - 1}\ dx = \int \sqrt{\dfrac{1-\cos x}{\cos x}}\ dx$

$$= \sqrt{2} \int \frac{\sin \frac{x}{2}}{\sqrt{2\cos^2 \frac{x}{2} - 1}}\ dx = -2\sqrt{2} \int \frac{dz}{\sqrt{2z^2 - 1}}$$

$$\left(\text{Putting } \cos\frac{x}{2} = z \Rightarrow \sin\frac{x}{2}\ dx = -2dz\right)$$

$$= -2 \int \frac{dz}{\sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -2 \log \left[z + \sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2} \right] + C$$

$$= -2 \log \left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}} \right) + C$$

Solve a tricky problem

Solve $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

Solution: $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

$$= \int \sqrt{\frac{\sin x}{\cos x \sin^2 x \cos^2 x}} dx$$

$$\int \frac{1}{\sqrt{\sin x \cos^3 x}} dx$$

$$\int \frac{1}{\sqrt{\sin^4 x \cot^3 x}} dx$$

$$-\int -cosec^2 x\ cot^{-3/2} x\ dx$$

$$= \frac{2}{\sqrt{\cot x}} + C$$

Solve another problem

$$I = \int \sqrt{1 + \operatorname{cosec} x}\ .\ dx$$

$$= \int \sqrt{1 + \frac{1}{\sin x}}\ .\ dx = \int \sqrt{\frac{\sin x + 1}{\sin x}}\ .\ dx$$

$$= \int \sqrt{\frac{(1+\sin x)(1-\sin x)}{\sin x\,(1-\sin x)}}\ .\ dx \qquad \text{[On rationalization]}$$

$$= \int \sqrt{\frac{1-\sin^2 x}{\sin x - \sin^2 x}}\ .\ dx \qquad [\because\ (a+b)(a-b) = a^2 - b^2]$$

$$= \int \frac{\cos x}{\sqrt{\sin x - \sin^2 x}}\ .\ dx \qquad [\because\ \sin^2 A + \cos^2 A = 1]$$

$$\sin x = z \quad \Rightarrow \quad \cos x\, dx = dz$$

$$I = \int \frac{1}{\sqrt{z - z^2}}\ .\ dz = \int \frac{1}{\sqrt{-(z^2 - z)}}\ .\ dz$$

$$= \int \frac{1}{\sqrt{\frac{1}{4} - \left(z^2 - z + \frac{1}{4}\right)}}\ .\ dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because\ \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(z - \frac{1}{2}\right)^2}}\ .\ dz$$

$$\left(z - \frac{1}{2}\right) = y \quad \Rightarrow \quad dz = dy$$

$$I = \int \frac{1}{\sqrt{(1/2)^2 - y^2}}\ .\ dy \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}}\ .\ dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \sin^{-1}\left(\frac{y}{1/2}\right) + c$$

$$= \sin^{-1}\left(\frac{z - 1/2}{1/2}\right) + c \qquad [\because\ y = z - 1/2]$$

Solve another Integral

$$I = \int \sqrt{\frac{1+x}{x}} \, . \, dx$$

$$= \int \sqrt{\frac{1+x}{x} \times \frac{1+x}{1+x}} \, dx \qquad \text{[Multiply and divided by } (1+x)]$$

$$= \int \sqrt{\frac{(1+x)^2}{x(1+x)}} \, . \, dx = \int \frac{1+x}{\sqrt{x+x^2}} \, . \, dx$$

Let us write :

$$1 + x = \lambda \, . \, \frac{d}{dx}(x + x^2) + \mu$$

$$\Rightarrow \quad 1 + x = \lambda (1 + 2x) + \mu \qquad ...(1)$$

$$\Rightarrow \quad 1 + x = 2\lambda x + \lambda + \mu$$

Comparing the coefficients of x and the constant terms, we have

$$1 = 2\lambda \quad \Rightarrow \quad \lambda = \frac{1}{2}$$

and

$$1 = \lambda + \mu \quad \Rightarrow \quad \mu = 1 - \lambda = 1 - \frac{1}{2} = \frac{1}{2} .$$

Putting the values of λ and μ in (1),

$$1 + x = \frac{1}{2}(1+2x) + \frac{1}{2} .$$

$$\therefore \quad I = \int \frac{\frac{1}{2}(1+2x) + \frac{1}{2}}{\sqrt{x+x^2}} \, . \, dx$$

$$= \frac{1}{2}\int \frac{1+2x}{\sqrt{x+x^2}} \, dx + \frac{1}{2}\int \frac{1}{\sqrt{x+x^2}} \, . \, dx$$

$$\Rightarrow \quad I = \frac{1}{2} \, I_1 + \frac{1}{2} \, I_2 \qquad ...(2)$$

Now

$$I_1 = \int \frac{1+2x}{\sqrt{x+x^2}} \, dx$$

Put

$$x + x^2 = z \quad \Rightarrow \quad (1 + 2x) \, dx = dz$$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \, . \, dz = \int z^{-1/2} \, . \, dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2\sqrt{z} + c_1$$

$$= 2\sqrt{x + x^2} + c_1 \qquad ...(3)$$

and

$$I_2 = \int \frac{1}{\sqrt{x+x^2}} \, . \, dx$$

$$= \int \frac{1}{\sqrt{\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4}}} \, . \, dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \, . \, dx$$

Put $x + \frac{1}{2} = z \Rightarrow dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{z^2 - \left(\frac{1}{2}\right)^2}} \, . \, dz \qquad \left[\text{By using } \int \frac{1}{\sqrt{x^2 - a^2}} \, . \, dx = \log\left| x + \sqrt{x^2 - a^2} \right| + c\right]$$

$$= \log\left| z + \sqrt{z^2 - \left(\frac{1}{2}\right)^2} \right| + c_2 = \log\left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}} \right| + c_2$$

$$= \log\left| \left(x + \frac{1}{2}\right) + \sqrt{x^2 + x} \right| + c_2 \qquad \ldots(4)$$

$\therefore$ **From equation (2),**

$$I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \text{[Using (3) and (4)]}$$

Solve another problem

$$I = \int \frac{ax^3 + bx}{x^4 + c^2}\, dx = \int \frac{ax^3}{x^4 + c^2} . dx + \int \frac{bx}{x^4 + c^2} . dx$$

$$= a \int \frac{x^3}{x^4 + c^2} . dx + b \int \frac{x}{x^4 + c^2} . dx$$

$\Rightarrow \quad I = a\, I_1 + b\, I_2$ ...(1)

Now $\quad I_1 = \int \frac{x^3}{x^4 + c^2} . dx$

$= \frac{1}{4} \int \frac{4x^3}{x^4 + c^2} . dx$ [Multiply and divided by 4]

$= \frac{1}{4} \log\left| x^4 + c^2 \right| + c_1$ ...(2) $\left[\because \int \frac{f'(x)}{f(x)} . dx = \log|f(x)| + c\right]$

and $\quad I_2 = \int \frac{x}{x^4 + c^2} . dx$

$= \frac{1}{2} \int \frac{2x}{(x^2)^2 + c^2}\, dx$ [Multiply and divided by 2]

Put $\quad x^2 = z \Rightarrow 2x\, dx = dz$

$= \frac{1}{2} \int \frac{1}{z^2 + c^2}\, dz$ $\left[\text{By using} \int \frac{1}{x^2 + a^2} . dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$

Solve Integration root linear plus root linear in denominator

If $I = \int \frac{dx}{\sqrt{2x+3} + \sqrt{x+2}}$, then I equals

(a) $2(u - v) + \log \left|\frac{u-1}{u+1}\right| + \log \left|\frac{v-1}{v+1}\right| + C$

$u = \sqrt{2x+3}$, $v = \sqrt{x+2}$

(b) $\log \left|\frac{\sqrt{x+2} + \sqrt{2x+3}}{\sqrt{x+2} - \sqrt{2x+3}}\right| + C$

(c) $\log \left(\sqrt{x+2} + \sqrt{2x+3}\right) + C$

(d) is transcedental function in u and v, $u = \sqrt{2x+3}$

$v = \sqrt{x+2}$

Ans. (a), (d)

$$I = \int \frac{\sqrt{2x+3} - \sqrt{x+2}}{x+1}\,dx$$

$$= I_1 - I_2$$

where $I_1 = \int \frac{\sqrt{2x+3}}{x+1}\,dx$ and $I_2 = \int \frac{\sqrt{x+2}}{x+1}\,dx$

Put $2x + 3 = t^2$, in I_1, so that

$$I_1 = \int \frac{2t \cdot t}{t^2 - 1}\,dt = 2\int\left[1 + \frac{1}{t^2-1}\right]dt$$

$$= 2\left[t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|\right]$$

In I_2, put $x + 2 = y^2$, so that

$$I_2 = \int \frac{2y^2}{y^2-1}\,dy = 2y + \log\left|\frac{y-1}{y+1}\right|$$

Thus,

$$I = 2\left(\sqrt{2x+3} - \sqrt{x+2}\right) + \log\left|\frac{\sqrt{2x+3}-1}{\sqrt{2x+3}+1}\right| + \log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$$

Solve another Problem

Evaluate $\int \frac{\sin 2x\, dx}{(a+b\cos x)^2}$.

Solution:

We have $I = \int \frac{\sin 2x\, dx}{(a+b\cos x)^2} = 2\int \frac{\sin x \cos x\, dx}{(a+b\cos x)^2}$

Now put $a + b \cos x = t$

so that $-b \sin x\, dx = dt$.

Also $\cos x = \frac{(t-a)}{b}$.

$$\therefore\ I = -\frac{2}{b}\int \frac{(t-a)/b}{t^2} dt = -\frac{2}{b^2}\int \left[\frac{t}{t^2} - \frac{a}{t^2}\right] dt$$

$$= -\frac{2}{b^2}\int \left[\frac{1}{t} - \frac{a}{t^2}\right] dt = -\frac{2}{b^2}\left[\log t + \frac{a}{t}\right]$$

$$= -\frac{2}{b^2}\left[\log(a+b\cos x) + \frac{a}{a+b\cos x}\right].$$

A special Integral

$$\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{(1+x+x^2)}}\,dx$$

Here we set $\sqrt{1+x+x^2} = xt + 1$, so that

$$x = \frac{2t-1}{1-t^2},\ dx = \frac{2t^2-2t+2}{(1-t^2)^2}\,dt \text{ and}$$

$$(1-\sqrt{1+x+x^2}) = \frac{-2t^2+t}{(1-t^2)}$$

Substitution of these values in the given integral transforms the problem in the form

$$\int \frac{(-2t^2+t)^2\,(1-t^2)^2\,(1-t^2)\,(2t^2-2t+2)}{(1-t^2)^2\,(2t-1)^2\,(t^2-t+1)\,(1-t^2)^2}\,dt$$

$$= +2\int \frac{t^2}{1-t^2}\,dt = -2t + \ln\left|\frac{1+t}{1-t}\right| + C$$

An advanced example

$$I = \int \frac{(x+1)}{x(1+xe^x)^2}\,dx$$

$$I = \int \frac{e^x(x+1)}{x\,e^x(1+xe^x)^2}\,dx$$

put $1 + xc^x = t$, $(xe^x + e^x)\,dx = dt$

$$I = \int \frac{dt}{(t-1)t^2} = \int \left(\frac{1}{1-t} + \frac{1}{t} + \frac{1}{t^2} \right) dt$$

$$= -\log|1-t| + \log|t| - \frac{1}{t} + C = \log\left|\frac{t}{1-t}\right| - \frac{1}{t} + C$$

$$= \log\left|\frac{1+xe^x}{-xe^x}\right| - \frac{1}{1+xe^x} + C = \log\left(\frac{1+xe^x}{xe^x}\right) - \frac{1}{1+xe^x} + C$$

Practice Example

Let $f(x)$ be a function defined by $f(x) = \int_1^x x(x^2 - 3x + 2)\,dx$, $1 \le x \le 3$, then the range of $f(x)$ is

(a) $\left[-\frac{1}{4}, 2\right]$ (b) $\left[-\frac{1}{4}, 4\right]$

(c) $[0, 2]$ (d) none of these

Solution :

(a). We have,

$f'(x) = x(x^2 - 3x + 2) = x(x-1)(x-2)$

Clearly, $f'(x) \le 0$ in $1 \le x \le 2$ and $f'(x) \ge 0$ in $2 \le x \le 3$.

$\therefore$ $f'(x)$ is monotonic decreasing in $[1, 2]$ and monotonic increasing in $[2, 3]$.

$$\therefore \text{Min. } f(x) = f(2) = \int_1^2 x(x^2 - 3x + 2)\,dx$$

$$= \left|\frac{x^4}{4} - x^3 + x^2\right|_1^2 = \frac{-1}{4}$$

Max. $f(x)$ = the greatest among $(f(1), f(3))$

Now, $f(1) = \int_1^1 x(x^2 - 3x + 2)\,dx = 0$

$$f(3) = \int_1^3 x(x^2 - 3x + 2)\,dx$$

$$= \frac{x^4}{4} - x^3 + 2\Big|_1^3 = 2. \quad \therefore \text{ Max. } f(x) = 2$$

Hence, Range $= \left[\frac{-1}{4}, 2\right]$

Practice Example

$$\int_{-2\pi}^{5\pi} \cot^{-1}(\tan x)\,dx$$

(a) $7\pi^2$ (b) $\frac{7\pi^2}{2}$

(c) 0 (d) $\frac{3\pi^2}{2}$

Solution :

(b). Let $I = \int_{-2\pi}^{5\pi} \cot^{-1}(\tan x)\,dx$

$$= 7\int_0^{\pi} \cot^{-1}\left(\cot(\pi/2 - x)\right)dx \qquad ...(1)$$

($\because$ Period is π)

Since $\cot^{-1}(\cot x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi + x, & \pi/2 < x < \pi \end{cases}$

$$\therefore \quad I = 7\left\{\int_0^{\pi/2}\left(\frac{\pi}{2} - x\right)dx + \int_{\pi/2}^{\pi}\left(\pi + \frac{\pi}{2} - x\right)dx\right\}$$

$$= 7\left\{\left(\frac{\pi}{2}x - \frac{x^2}{2}\right)_0^{\pi/2} + \left(\frac{3\pi}{2}x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right\}$$

$$= 7\left\{\left(\frac{\pi^2}{4} - \frac{\pi^2}{8}\right) + \left(\frac{3\pi^2}{2} - \frac{\pi^2}{2} - \frac{3\pi^2}{4} + \frac{\pi^2}{8}\right)\right\}$$

$$= \frac{7\pi^2}{2}$$

Practice Example

$f(x)$ is a continuous function for all real values of x and satisfies $\int_0^x f(t)\,dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^6}{3} + k$. The value of k is

(a) $\frac{167}{840}$ (b) $-\frac{167}{840}$

(c) $\frac{17}{38}$ (d) none of these

Solution :

(b). We have,

$$\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^6}{3} + k \quad ...(1)$$

For $x = 1$, $\int_x^1 f(t)dt = 0 + \frac{1}{8} + \frac{1}{3} + k = \frac{11}{24} + k \quad ...(2)$

Differentiating both sides of (1), w.r.t. x, we get

$$f(x) = -x^2 f(x) + 2x^{15} + 2x^5$$

$$\Rightarrow \quad f(x) = \frac{2(x^{15} + x^5)}{1 + x^2}$$

$$\therefore \quad \int_0^1 f(t)dt = 2\int_0^1 \frac{(t^{15} + t^5)}{1 + t^2} = \frac{11}{24} + k \quad \text{(using (2))}$$

$$\Rightarrow \quad 2\int_0^1 (t^{13} - t^{11} + t^9 - t^7 + t^5)dt = \frac{11}{24} + k$$

$$\Rightarrow \quad 2\left(\frac{1}{14} - \frac{1}{12} + \frac{1}{10} - \frac{1}{8} + \frac{1}{6}\right) = \frac{11}{24} + k$$

$$\Rightarrow \quad k = -\frac{167}{840}$$

Practice Example

If $I = \int_{-\pi}^{\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}}\, dx \qquad (1)$

then I equals

(a) 2π (b) π

(c) $\pi/2$ (d) $\pi/4$

Ans. (b)

Solution Using $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$,

we get

$$I = \int_{-\pi}^{\pi} \frac{e^{\sin(-x)}}{e^{\sin(-x)} + e^{-\sin(-x)}} dx$$

$$\Rightarrow \quad I = \int_{-\pi}^{\pi} \frac{e^{-\sin x}}{e^{-\sin x} + e^{\sin x}} dx \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_{-\pi}^{\pi} \frac{e^{\sin x} + e^{-\sin x}}{e^{\sin x} + e^{-\sin x}} dx = 2\pi$$

$$\Rightarrow \quad I = \pi.$$

Practice Example

If $I = \int_0^a \sqrt{\frac{a-x}{a+x}}\, dx$, $a > 0$, then I equals

(a) $\frac{1}{2}\left(a - \frac{\pi}{2}\right)$ (b) $\frac{a}{2}(\pi - 1)$

(c) $\frac{1}{\sqrt{2}} a(\pi - 1)$ (d) $a\left(\frac{\pi}{2} - 1\right)$

Ans. (d)

Solution We can write

$$I = \int_0^a \frac{a-x}{\sqrt{a^2 - x^2}}\, dx$$

$$= \left[a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}\right]_0^a$$

$$= a\left(\frac{\pi}{2} - 1\right).$$

Practice Example

If $f(x)=\dfrac{x-1}{x+1}$, $f^2(x)=f(f(x)),\ldots f^{k+1}_{(x)}=f(f^k(x)), k=1$

$2, 3, \ldots$ and $\phi(x)=f^{1998}(x)$, then $\int_{1/e}^{1}\phi(x)\,dx=$

(a) 1 (b) -1

(c) 0 (d) none of these

Solution :

(b). We have, $f(x)=\dfrac{x-1}{x+1}$

$$\Rightarrow \quad f^2(x)=f(f(x))=f\left(\frac{x-1}{x+1}\right)=\frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1}=-\frac{1}{x}$$

$$\Rightarrow \quad f^4(x)=f^2(f^2(x))=f^2\left(-\frac{1}{x}\right)=\frac{-1}{\frac{-1}{x}}=x$$

$$\therefore \quad \phi(x)=f^{1998}(x)=f^2(f^{1996}(x))=f^2(x)$$

$$\left[\because\ f^{1996}(x)=\frac{(f^4(f^4(f^4\ldots f^4)(x)}{499\text{ times}}=x\right]$$

$$\Rightarrow \quad \phi(x)=-\frac{1}{x}.$$

$$\therefore \quad \int_{1/e}^{1}\phi(x)\,dx=\int_{1/e}^{1}\left(-\frac{1}{x}\right)dx=(\log_e x)\Big|_{1/e}^{1}$$

$$=-(\log_e 1-\log_e 1/e)=-(0+1)=-1$$

Practice Example

If I =

$$\int_0^{\pi} e^{|(1/2)\cos x|}\left\{2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right\}\sin x\ dx$$

then I equals

(a) $7\sqrt{e}\cos(1/2)$ (b) $7\sqrt{e}[\cos(1/2)-\sin(1/2)]$

(c) 0 (d) none of these

Ans. (d)

Solution Put $\frac{1}{2}\cos x = t$, so that $-\sin x\, dx = 2dt$ and

$$I = \int_{1/2}^{-1/2} e^{|t|}(2\sin t + 3\cos t)(-2)\, dt$$

As $e^{|t|}\sin t$ is an odd function, and $e^{|t|}\cos t$ is an even function,

$$I = 6\int_0^{1/2} e^t\cos t\, dt = 6e^t\cos t\Big]_0^{1/2} + 6\int_0^{1/2} e^t\sin t\, dt$$

$$I = 6\left[\sqrt{e}\cos\left(\frac{1}{2}\right)-1\right] + 6e^t\sin t\Big]_0^{1/2} - 6\int_0^{1/2} e^t\cos t\, dt$$

$$\Rightarrow 7I = 6\sqrt{e}\left(\cos\left(\frac{1}{2}\right)+\sin\left(\frac{1}{2}\right)-1\right)$$

Practice Example

$\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n}\sin^{2k}\frac{r\pi}{2n}$ is equal to

(a) $\frac{2k!}{2^{2k}\,(k!)^2}$ (b) $\frac{2k!}{2^{k}\,(k!)}$

(c) $\frac{2k!}{2^{k}\,(k!)^2}$ (d) none of these

Solution :

(a). $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n}\sin^{2k}\frac{r\pi}{2n}$

$$=\int_0^1 \sin^{2k}\frac{\pi x}{2}\,dx=\frac{2}{\pi}\int_0^{\pi/2}\sin^{2k} t\,dt$$

$$\left[\text{Putting } \frac{\pi x}{2}=t \Rightarrow dx=\frac{2}{\pi}dt\right]$$

$$=\frac{2}{\pi}\cdot\frac{(2k-1)(2k-3)\cdots\cdots 1}{2k(2k-2)\cdots\cdots 2}\cdot\frac{\pi}{2}$$

$$=\frac{[(2k-1)(2k-3)(2k-5)\cdots 1][2k\cdot(2k-2)\cdots\cdots 2]}{2^k[k(k-1)(k-2)\cdots 1][2k\cdot(2k-2)\cdots 2]}$$

$$=\frac{2k(2k-1)(2k-2)(2k-3)\cdots 2\cdot 1}{2^k[k(k-1)(k-2)\cdots 1]\cdot 2^k[k\cdot(k-1)(k-2)\cdots 1]}$$

$$=\frac{(2k)!}{2^{2k}\cdot(k!)^2}.$$

Practice Example

$$\text{If } I_1 = \int_0^{\pi/2} f(\sin 2x) \sin x \, dx$$

and $I_2 = \int_0^{\pi/4} f(\cos 2x) \cos x \, dx$,

then $\frac{I_1}{I_2}$ equals

(a) 1 (b) $1/\sqrt{2}$

(c) $\sqrt{2}$ (d) 2

Ans. (c)

Solution Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$, we get

$$I_1 = \int_0^{\pi/2} f[\sin(\pi - 2x)] \sin(\pi/2 - x)\, dx$$

$$= \int_0^{\pi/2} f(\sin 2x) \cos x \, dx \qquad (2)$$

Adding (1) and (2) we get

$$2I_1 = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\, dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x) \cos\left(x - \frac{\pi}{4}\right) dx$$

Put $x - \pi/4 = \theta$, so that

$$2I_1 = \sqrt{2}\int_{-\pi/4}^{\pi/4} f[\sin(\pi/2 + 2\theta)] \cos\theta \, d\theta$$

$$= \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2\theta) \cos\theta \, d\theta$$

$= 2\sqrt{2}\, I_2$ as in integrand is an even function

$\Rightarrow I_1/I_2 = \sqrt{2}$.

Practice Example

If $I = \int_{-1}^{2} |x \sin \pi x|\, dx$, then I equals

(a) $1/\pi$ (b) $2/\pi$
(c) $4/\pi$ (d) $5/\pi$

Ans. (d)

Solution We can write

$$I = \int_{-1}^{1} |x \sin \pi x|\, dx + \int_{1}^{2} |x \sin \pi x|\, dx$$

As $|x \sin \pi x|$ is an even function, $\sin \pi x \geq 0$, for $0 \leq \pi x \leq \pi$ and $\sin \pi x \leq 0$ for $\pi \leq \pi x \leq 2\pi$, we get

$$I = 2 \int_{0}^{1} x \sin \pi x\, dx - \int_{1}^{2} x \sin \pi x\, dx$$

But $\int x \sin \pi x\, dx = \frac{-x \cos \pi x}{\pi} + \frac{1}{\pi}\int \cos \pi x\, dx$

$$= -\frac{x}{\pi} \cos \pi x + \frac{1}{\pi^2} \sin \pi x$$

Thus,

$$I = 2\left(-\frac{1}{\pi}\cos \pi + 0\right) - \left(-\frac{2}{\pi}\cos 2\pi + \frac{1}{\pi}\cos \pi\right)$$

$$= \frac{5}{\pi}$$

Practice Example

If $f(x) = \int_0^x (1+t^3)^{-1/2}\,dt$ and g is the inverse of f, then the value of $\frac{g''}{g^2}$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$

(c) 1 (d) cannot be determined

Solution :

(b). We have,

$$f(x) = \int_0^x (1+t^3)^{-1/2}\,dt$$

$$\Rightarrow \quad f(g(x)) = \int_0^{g(x)} (1+t^3)^{-1/2}\,dt$$

$$\Rightarrow \quad x = \int_0^{g(x)} (1+t^3)^{-1/2}\,dt$$

[g is inverse of $f \Rightarrow f\{g(x)\} = x$]

Differentiating w.r.t. x, we have

$$1 = (1+g^3)^{-1/2} \cdot g'$$

i.e., $(g')^2 = 1 + g^3$

Differentiating again w.r.t. x, we have

$$2g'g'' = 3g^2g'$$

$$\Rightarrow \quad \frac{g''}{g^2} = \frac{3}{2}.$$

Practice Example

Let $f(x) = \dfrac{|x|}{x}$ if $x \neq 0$ and $f(0) = 0$ and $a, b \in \mathbf{R}$ be such that $a < b$. Then value of

$$I = \int_a^b f(x)\, dx \text{ is}$$

(a) $|b| - |a|$ (b) $\frac{1}{2}(b^2 - a^2)$

(c) Max $\{|a|, |b|\}$ (d) Min $\{|a|, |b|]$

Ans. (a)

Solution Note that

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

If $0 \leq a < b$, then

$$I = \int_a^b dx = b - a = |b| - |a|$$

If $a < 0 \leq b$, then

$$I = \int_a^b (-1)\, dx + \int_0^b 1\, dx = a + b = b - (-a)$$
$$= |b| - |a|$$

If $a < b < 0$, then

$$I = \int_a^b (-1) dx = -b + a = |b| - |a|.$$

Practice Example

If $I_n = \int_0^{\pi/2} \cos^n x \cos nx\, dx$, then I_1, I_2, I_3 are in

(a) A. P. (b) G. P.

(c) H. P. (d) none of these

Solution :

(b). $I_n = \int_0^{\pi/2} \cos^n x \cos nx\, dx$

$= \left[\cos^n x \cdot \frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} n \cos^{n-1} x(-\sin x) \cdot \frac{\sin nx}{n} dx$

$= 0 + \int_0^{\pi/2} \cos^{n-1} x \sin x \sin nx\, dx$

$= \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x - \int_0^{\pi/2} \cos^n x \cos nx\, dx$

[Using the identity

$\cos(n-1)x = \cos nx \cos x + \sin nx \sin x$

i.e., $\sin nx \sin x = \cos(n-1)x - \cos nx \cos x$]

$= \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x - \int_0^{\pi/2} \cos^n x \cos nx\, dx$

$= I_{n-1} - I_n$

i.e., $\frac{I_n}{I_{n-1}} = \frac{1}{2} \Rightarrow I_1, I_2, I_3$ are in G.P

Practice Example

$\int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right] dx = A. \frac{\sin k \sin\left(k+\frac{1}{2}\right)}{\sin\frac{1}{2}}$, where A is equal to

(a) π (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) none of these

Solution :

(c). We have,

$$\int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right] dx$$

$$= \int_0^{\pi/2} \sin 0\, dx + \int_{\pi/2}^{2\pi/2} \sin 1\, dx + \int_{2\pi/2}^{3\pi/2} \sin 2\, dx + \ldots + \int_{(2k-1)\pi/2}^{2k\pi/2} \sin(2k-1) dx$$

$$= \frac{\pi}{2}[\sin 1 + \sin 2 + \sin 3 + \ldots + \sin(2k-1)]$$

$$= \frac{\frac{\pi}{2}\left[\sin\frac{1}{2}\sin 1 + \sin\frac{1}{2}\sin 2 + \sin\frac{1}{2}\sin 3 + \ldots + \sin\frac{1}{2}\sin(2k-1)\right]}{\sin\frac{1}{2}}$$

$$= \frac{\frac{\pi}{2}\left[\begin{array}{l}\cos\frac{1}{2} - \cos\frac{3}{2} + \cos\frac{3}{2} - \cos\frac{5}{2} + \ldots + \\ \cos\left(2k - \frac{3}{2}\right) - \cos\left(2k + \frac{1}{2}\right)\end{array}\right]}{2\sin\frac{1}{2}}$$

$$= \frac{\frac{\pi}{2}\left(\cos\frac{1}{2} - \cos\left(2k + \frac{1}{2}\right)\right)}{2\sin\frac{1}{2}}$$

$$= \frac{\pi}{2} \cdot \frac{\sin k \,.\, \sin\left(k + \frac{1}{2}\right)}{\sin\left(\frac{1}{2}\right)} \qquad \therefore A = \pi/2.$$

Practice Example

For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1+t}\,dt$. Then, the value of $f(e) + f\left(\frac{1}{e}\right)$ is

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) none of these.

Solution :

(c). We have,

$$f(x) = \int_1^x \frac{\ln t}{1+t}\,dt,\ x>0 \qquad \text{...(1)}$$

$$\Rightarrow \qquad f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1+t}\,dt$$

Put $y = \frac{1}{t} \Rightarrow dt = \frac{-1}{y^2}\,dy$

$$\therefore \qquad f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln\left(\frac{1}{y}\right)}{1+\frac{1}{y}}\left(\frac{-1}{y^2}\right)dy$$

$$= \int_1^x \frac{\ln y}{y(1+y)}\,dy$$

$$= \int_1^x \frac{\ln t}{(1+t)t}\,dt \qquad \text{...(2)}$$

From (1) and (2),

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\left(1+\frac{1}{t}\right)\ln t}{1+t}\,dt$$

$$= \int_1^x \frac{\ln t}{t}\,dt = \frac{(\ln x)^2}{2}$$

$$\Rightarrow \quad f(e) + f\left(\frac{1}{e}\right) = \frac{(\ln e)^2}{2} = \frac{1}{2}.$$

Practice Example

If $I_n = \int_0^1 (1-x^a)^n \, dx$, then $\frac{I_n}{I_{n+1}} = 1 + \frac{1}{k}$, where $k =$

(a) $(n+1)a$ (b) na

(c) $(n-1)a$ (d) none of these

Solution :

(a). We have,

$$I_{n+1} = \int_0^1 (1-x^a)^{n+1} \, dx$$

$$= \left[x(1-x^a)^{n+1}\right]_0^1 + (n+1)a \int_0^1 x^a (1-x^a)^n \, dx$$

$$= (n+1)a \int_0^1 (x^a - 1 + 1)(1-x^a)^n dx$$

$$= (n+1)a \int_0^1 (1-x^a)^n dx - (n+1)a \int_0^1 (1-x^a)^{n+1} \, dx$$

$$= (n+1)a\, I_n - (n+1)\, a\, I_{n+1}$$

$$\Rightarrow \quad \frac{I_n}{I_{n+1}} = 1 + \frac{1}{(n+1)a} \quad \therefore k = (n+1)a$$

Practice Example

If $I = \int_{\alpha}^{\beta}\left[\log\log x + \frac{1}{(\log x)^2}\right] dx$, then I equals

(a) $\alpha \log\log \alpha - \beta \log\log \beta$

(b) $\frac{1}{\alpha} - \frac{1}{\beta} + \log\log \alpha - \log\log \beta$

(c) $\frac{\beta-\alpha}{\alpha\beta} + \alpha \log\log \alpha - \beta \log\log \beta$

(d) none of these

Ans. (d)

Solution Put $\log x = t$, or $x = e^t$, so that

$$I = \int_a^b \left[\log t + \frac{1}{t^2}\right] e^t \, dt$$

where $a = \log \alpha$, $b = \log \beta$

$$= \int_a^b \left(\log t + \frac{1}{t} + \left(-\frac{1}{t}\right) + \frac{1}{t^2}\right) e^t \, dt$$

$$= \left(\log t - \frac{1}{t}\right)e^t \Big]_a^b$$

[use $\int e^x (f(x) + f'(x)) = e^x f(x)$]

$$= \left(\log b - \frac{1}{b}\right)e^b - \left(\log a - \frac{1}{a}\right)e^a$$

$$= \left(\log\log\beta - \frac{1}{\log\beta}\right)\beta - \left(\log\log\alpha - \frac{1}{\log\alpha}\right)\alpha$$

Practice Example

Let $\phi(x) = \int_0^x g(t)\,dt$, where the function g is such that

$-\frac{1}{2} \le g(t) \le 0, \ \forall \ t \in [0, 1]$

$\frac{1}{2} \le g(t) \le 1, \ \forall \ t \in [1, 3]$

$g(t) \le 1, \ \forall \ t \in [3, 4]$

Then, $\phi(4)$ satisfies the inequality

(a) $\frac{1}{2} \le \phi(4) \le 3$ (b) $0 \le \phi(4) \le 2$

(c) $\phi(4) \le 3$ (d) none of these

Solution :

(c). We have,

$$\phi(4) = \int_0^4 g(t)\,dt = \int_0^1 g(t)\,dt + \int_1^3 g(t)\,dt + \int_3^4 g(t)\,dt$$

But

$$\frac{-1}{2}\cdot 1 \le \int_0^1 g(t)\,dt \le 0.1$$

$$\frac{1}{2}\cdot 2 \le \int_1^3 g(t)\,dt \le 0.2$$

$$\int_3^4 g(t)\,dt \le 1.1$$

Adding the above inequalities, we get $\phi(4) \le 3$

Practice Example

If $I = \int_0^\infty \frac{\sqrt{x}\,dx}{(1+x)(2+x)(3+x)}$, then I equals

(a) $\frac{\pi}{2}(2\sqrt{2}-\sqrt{3}-1)$ (b) $\frac{\pi}{2}(2\sqrt{2}+\sqrt{3}-1)$

(c) $\frac{\pi}{2}(2\sqrt{2}-\sqrt{3}+1)$ (d) none of these

Ans. (a)

Solution Put $\sqrt{x} = t$ or $x = t^2$, so that

$$I = 2\int_0^\infty \frac{t^2}{(1+t^2)(2+t^2)(3+t^2)}\,dt$$

$$= \int_0^\infty \left(-\frac{1}{1+t^2} + \frac{4}{2+t^2} - \frac{3}{3+t^2}\right) dt$$

$$= \left[-\tan^{-1} t + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_0^\infty$$

$$= -\frac{\pi}{2} + 2\sqrt{2}\left(\frac{\pi}{2}\right) - \sqrt{3}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}(2\sqrt{2}-\sqrt{3}-1).$$

Practice Example

$\int_1^4 (\{x\})^{[x]}\,dx$, where $\{\cdot\}$ and $[\cdot]$ denote the fractional part and greatest integer function, respectively, is equal to

(a) 1 (b) $\frac{12}{13}$

(c) $\frac{13}{12}$ (d) $\frac{6}{7}$

Solution :

(c). We have,

$$\int_1^4 (\{x\})^{[x]}\,dx$$

$$= \int_1^4 (x-[x])^{[x]}\,dx$$

$$= \int_1^2 (x-[x])^{[x]}\,dx + \int_2^3 (x-[x])^{[x]}\,dx + \int_3^4 (x-[x])^{[x]}\,dx$$

$$= \int_1^2 (x-1)^1\,dx + \int_2^3 (x-2)^2\,dx + \int_3^4 (x-3)^3\,dx$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^2 + \left[\frac{(x-2)^3}{3}\right]_2^3 + \left[\frac{(x-3)^4}{4}\right]_3^4$$

$$= \left(\frac{1}{2}-0\right)+\left(\frac{1}{3}-0\right)+\left(\frac{1}{4}-0\right) = \frac{13}{12}.$$

Practice Example

The value $\int_0^1 \cot^{-1}(1 + x^2 - x)\, dx$ is

(a) $\pi/2 - \log 2$ (b) $\pi - \log 2$

(c) $\pi/4 - \log 2$ (d) $2\int_0^1 \tan^{-1} x\, dx$

Ans. (a), (d)

Solution $\cot^{-1}(1 + x^2 - x) = \tan^{-1}\left(\dfrac{x + 1 - x}{1 - x(1 - x)}\right)$

$$= \tan^{-1} x + \tan^{-1}(1 - x)$$

$$I = \int_0^1 \cot^{-1}(1 + x^2 - x)dx = \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(1 - x)dx$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1} x\,dx = 2\int_0^1 \tan^{-1} x\,dx$$

$$= 2x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{2x}{1 + x^2}dx$$

$$= 2\tan^{-1}(1) - \log(1 + x^2)\Big]_0^1$$

$$= 2(\pi/4) - \log 2 = \pi/2 - \log 2$$

Practice Example

If $[\cdot]$ denotes the greatest integer function, then

$$\int_0^2 [x+[x+[x]]]dx =$$

(a) 1 (b) 2

(c) 3 (d) 0

Solution :

$$I = \int_0^2 [x+[x+[x]]]dx$$

$$= \int_0^2 [x+2[x]]dx \ (\because [x+\text{Integer}] = [x] + \text{Integer} \Rightarrow [x+[x]] = [x]+[x])$$

$$= \int_0^2 [x]+2[x]dx = \int_0^2 3[x]dx$$

$$= 3\left\{\int_0^1 [x]dx + \int_1^2 [x]dx\right\}$$

$$= 3\left\{\int_0^1 0.dx + \int_1^2 1dx\right\}$$

$$= 3\ \{(x)_1^2\} = 3(2-1) = 3.$$

Practice Example

The value of $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$ is

(a) $\left(\int_{\pi}^{5\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx\right)^2$ (b) $\pi^2/16$

(c) $3\pi^2/4$ (d) $\pi^2/2$

Solution

Ans. (a), (b)

Solution

$$I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int_0^{\pi/2} \frac{(\pi/2 - x) \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx - \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{4} \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sin 2x}{1 - \frac{1}{2} \sin^2 2x} = \frac{\pi}{4} \int_0^{\pi/2} \frac{\sin 2x}{1 + \cos^2 2x} dx$$

$$= \frac{-\pi}{8} \int_1^{-1} \frac{dt}{1+t^2} = \frac{-\pi}{8} [\tan^{-1}(-1) - \tan^{-1} 1] = \frac{\pi^2}{16}$$

The integrand in a is a periodic function with period π, since

$$f(x + \pi) = \frac{\sin 2(x+\pi)}{\cos^4 (x+\pi) + \sin^4 (x+\pi)}$$

$$= \frac{\sin 2x}{\cos^4 x + \sin^4 x} = f(x)$$

$$\therefore \quad \int_{\pi}^{5\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx = \int_0^{\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx$$

$$= 2\int_0^{\pi/4} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

$$= \int_0^1 \frac{2t}{1+t^4} dt = \tan^{-1} t^2 \Big|_0^1 = \frac{\pi}{4}$$

Practice Example

$$\int_{\sqrt{(3a^2+b^2)/4}}^{\sqrt{(a^2+b^2)/2}} \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx =$$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{12}$

Solution :

(d). Let $I = \int_{\sqrt{(3a^2+b^2)/4}}^{\sqrt{(a^2+b^2)/2}} \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx$

Put $x^2 = a^2\cos^2 t + b^2\sin^2 t$

$\Rightarrow \quad 2x\,dx = [2a^2\cos t(-\sin t) + 2b^2\sin t(\cos t)]\,dt$

$\Rightarrow \quad x\,dx = \frac{1}{2}(b^2-a^2)\sin 2t.dt$

For $\quad x^2 = \frac{a^2+b^2}{2} = a^2\cos^2 t + b^2\sin^2 t$

$\Rightarrow \quad a^2+b^2 = 2(1-\sin^2 t)a^2 + 2b^2\sin^2 t$

or, $\quad (a^2+b^2) = 2a^2 + 2(b^2-a^2)\sin^2 t$

$\Rightarrow \quad \sin^2 t = \frac{1}{2} \Rightarrow \cos 2t = 0 \quad \Rightarrow t = \pi/4$

For $\quad x^2 = \frac{3a^2+b^2}{4} = a^2\cos^2 t + b^2\sin^2 t$

$\Rightarrow \quad 3a^2+b^2 = 4a^2 + 4(b^2-a^2)\sin^2 t$

$\Rightarrow \quad \sin^2 t = \frac{1}{4} \Rightarrow \cos 2t = \frac{1}{2} \Rightarrow t = \frac{\pi}{4}$

$$\therefore \quad I = \int_{\pi/6}^{\pi/4} \frac{1}{2}\frac{(b^2-a^2)\sin 2t.dt}{\sqrt{(b^2-a^2)\sin^2 t(b^2-a)\cos^2 t}}$$

$$= \int_{\pi/6}^{\pi/4} dt = (t)_{\pi/6}^{\pi/4} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

Practice Example

If $\int_0^{\pi/2} \frac{x^2 \cos x}{(1+\sin x)^2}\,dx = A\,\pi - \pi^2$ then A is

Ans. 2

Solution Integrating by parts, we have

$$\int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2}\,dx$$

$$= -\left.\frac{x^2}{1+\sin x}\right|_0^{\pi} + 2\int_0^{\pi} \frac{x}{1+\sin x}\,dx = -\pi^2 + 2I$$

where

$$I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx = \pi\int_0^{\pi} \frac{dx}{1+\sin x} - I$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{dx}{1+\sin x} = 2\pi\int_0^{\pi/2} \frac{dx}{1+\sin x}$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2} \frac{dx}{1+\sin x} = \pi\int_0^{\pi/2} \frac{dx}{1+\sin(\pi/2 - x)}$$

$$= \int_0^{\pi/2} \frac{dx}{1+\cos x}$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \sec^2(x/2)\,dx = \pi \tan(x/2)\Big]_0^{\pi/2} = \pi$$

Hence $\int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2}\,dx = -\pi^2 + 2\pi$

Practice Example (CBSE 2010)

Evaluate: $\int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\ dx$ [CBSE 2010,4 marks]

Soln.:

Let sin x-cos x= t.......(i)

Differentiating, cos x-(-sin x) dx=dt

Or, (cos x+sin x)dx= dt

Also,

Squaring (i),

$sin^2 x + cos^2 x - 2\sin x\cos x = t^2$

Or, 1-2 sin x cos x= t^2

Or, 1-sin 2x=t^2

Or, sin 2x=1-t^2

Therefore, I= $\int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\ dx$

$= \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

(Since, when x= $\pi/6$, t=$\frac{1}{2}-\frac{\sqrt{3}}{2}=\frac{1-\sqrt{3}}{2}$ and when x= $\pi/3$, t=$\frac{\sqrt{3}}{2}-\frac{1}{2}=\frac{\sqrt{3}-1}{2}$)

$=[\ sin^{-1}\ t\ ]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$

$= sin^{-1}\ \frac{\sqrt{3}-1}{2} - sin^{-1}\ \frac{1-\sqrt{3}}{2}$

$= sin^{-1}\ \frac{\sqrt{3}-1}{2} + sin^{-1}\ \frac{\sqrt{3}-1}{2}$

$= 2\ sin^{-1}\ \frac{\sqrt{3}-1}{2}$ Ans.

Practice example

Integration Sin n plus half by Sin x by 2

The value of the integral $\int_0^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)}\,dx \ (n \in N)$ is

(a) π (b) 2π

(c) 3π (d) none of these

Ans. (a)

Solution We have, $2\sin\frac{x}{2}\left(\frac{1}{2}+\cos x+\cos 2x+\ldots+\cos nx\right)$

$$= \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + 2\sin\frac{x}{2}\cos 2x + \cdots + 2\sin\frac{x}{2}\cos nx$$

$$= \sin\frac{x}{2} + \sin\frac{3x}{2} - \sin\frac{x}{2} + \sin\frac{5x}{2} - \sin\frac{3x}{2} + \cdots$$

$$+ \sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x = \sin\left(n+\frac{1}{2}\right)x$$

$$\therefore \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin(x/2)}$$

$$\Rightarrow \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin(x/2)}\,dx = 2\left(\int_0^{\pi}\frac{1}{2}\,dx + \int_0^{\pi}\cos x\,dx + \ldots + \int_0^{\pi}\cos nx\,dx\right)$$

$$= 2\left(\frac{\pi}{2} + \sin x\Big|_0^{\pi} + \ldots + \frac{\sin nx}{n}\Big|_0^{\pi}\right) = \pi$$

Practice example

Example $\int_0^{\pi} \frac{dx}{(1+a^2)-2a\cos x} = \frac{\pi}{1-a^2}$ or $\frac{\pi}{a^2-1}$

according as $a < 1$ or $a > 1$.

The given problem may be re-written in the form

$$\int_0^{\pi} \frac{dx}{(1+a^2)\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right) - 2a\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

which can be expressed in the forms

$$I = \frac{2}{(1+a^2)^2}\int \frac{dt}{\left(\frac{1-a}{1+a}\right)^2 + t^2} \text{ or } \frac{2}{(1+a^2)^2}\int \frac{dt}{\left(\frac{a-1}{a+1}\right)^2 + t^2}$$

according as $a < 1$ or > 1, where $t = \tan\frac{x}{2}$

Hence

Hence

$$I = \frac{2}{(1-a^2)}\left[\tan^{-1}\frac{t(1+a)}{(1-a)}\right]_0^{\infty} = \frac{\pi}{1-a^2} \text{ if } a < 1$$

Similarly in the other case the answer shall be $\frac{\pi}{a^2-1}$, $a > 1$

Practice example

$\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})\,dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})\,dt$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) 0 (d) none of these

Solution :

(a). We have,

$$I = \int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})\,dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})\,dt$$

$$= \left[t\sin^{-1}(\sqrt{t})\right]_0^{\sin^2 x} - \int_0^{\sin^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$$

$$+ \left[t\cos^{-1}(\sqrt{t})\right]_0^{\cos^2 x} - \int_0^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$$

$$= x\sin^2 x + \int_{\sin^2 x}^{0} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt + x\cos^2 x + \int_0^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$$

$$= x(\sin^2 x + \cos^2 x) + \int_{\sin^2 x}^{\cos^2 x} \frac{\sqrt{t}}{\sqrt[2]{1-t}}\, dt$$

Putting $t = \sin^2\theta$ and $dt = 2\sin\theta\cos\theta\, d\theta$, we get,

$$\int \frac{\sqrt{t}}{\sqrt[2]{1-t}}\, dt = \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{\theta}{2} - \frac{\sin 2\theta}{4}$$

Also, when $t = \sin^2 x$, $\theta = x$ and when $t = \cos^2 x$, $\theta = \pi/2 - x$

$$\therefore\ I = x + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_x^{\pi/2 - x}$$

$$= x + \left(\frac{\pi}{4} - \frac{x}{2} - \frac{\sin 2x}{4}\right) - \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)$$

$$= x + \frac{\pi}{4} - x = \frac{\pi}{4}$$

Practice example

$$I = \int_0^{\pi/4} \frac{\sin 2\theta\, d\theta}{\sin^4\theta + \cos^4\theta} = \int_0^{\pi/4} \frac{2\sin\theta\cos\theta}{\sin^4\theta + \cos^4\theta}\, d\theta$$

$$= \int_0^{\pi/4} \frac{2\tan\theta\sec^2\theta\, d\theta}{1 + \tan^4\theta},$$

dividing the numerator and denominator by $\cos^4\theta$

Put $\tan^2\theta = t$,

so that $2\tan\theta\sec^2\theta\, d\theta = dt$.

When $\theta = 0$,

$t = \tan^2 0 = 0$

and when $\theta = \frac{\pi}{4}$,

$$t = \tan^2 \frac{1}{4}\pi = 1.$$

$$\therefore \quad I = \int_0^1 \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Practice example

If $f(x)$ satisfies the relation $\int_{-2}^{x} f(t)\, dt + x f'''(3)$

$= \int_{1}^{x} x^3 dx + f'(1) \int_{2}^{x} x^2 dx + f''(2) \int_{3}^{x} x\, dx$, then

(a) $f(x) = x^3 + 5x^2 + 2x - 6$

(b) $f(x) = x^3 - 5x^2 + 2x + 6$

(c) $f(x) = x^3 + 5x^2 + 2x - 6$

(d) $f(x) = x^3 - 5x^2 + 2x - 6$

Solution :

(d). Differentiating the given equation w.r.t. x, we get

$$f(x) + f'''(3) = x^3 + x^2 f'(1) + x f''(2) \qquad ...(1)$$

Differentiating successively w.r.t. x, we get

$$f'(x) = 3x^2 + 2x f'(1) + f''(2) \qquad ...(2)$$

$$f''(x) = 6x + 2f'(1) \qquad ...(3)$$

$$f'''(x) = 6 \qquad ...(4)$$

Putting $x = 1$, 2 and 3 in equations (2), (3) and (4) respectively, we get

$$f'(1) = 3 + 2f'(1) + f''(2), \quad f''(2) = 12 + 2f'(1)$$

and, $f'''(3) = 6$

Solving, we have

$$f'(1) = -5, f''(2) = 2, f'''(3) = 6$$

Putting the values in equation (1), we have

$$f(x) = x^3 - 5x^2 + 2x - 6.$$

Practice example

If $I_1 = \int_{1/e}^{\tan x} \frac{t}{1+t^2}\, dt$ and $I_2 = \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$ then the value of $I_1 + I_2$ is

(a) 1/2 (b) 1
(c) $e/2$ (d) $(1/2)(e + 1/e)$

Ans. (b)

Solution Putting $t = 1/u$ in I_2 we have

$$I_2 = -\int_{e}^{\tan x} \frac{u\,du}{1+u^2} = -\int_{1/e}^{\tan x} \frac{u\,du}{1+u^2} + \int_{1/e}^{e} \frac{u\,du}{1+u^2}$$

$$= -I_1 + \frac{1}{2}\int_{1/e}^{e} \frac{2u\,du}{1+u^2}$$

So $$I_1 + I_2 = \frac{1}{2}\log(u^2+1)]_{1/e}^{e} = \frac{1}{2}\left[\log(e^2+1) - \log\left(\frac{e^2+1}{e^2}\right)\right]$$

$$= \frac{1}{2} \times 2 = 1.$$

Practice example

$\lim_{x\to 0}\frac{1}{x}\left[\int_0^{x+y} e^{\sin^2 t}dt - \int_0^{y} e^{\sin^2 t}dt\right]$, where y is a constant independent of x, is equal to

(a) $e^{\sin^2 y}$ (b) $2e^{\sin^2 y}$

(c) $-e^{\sin^2 y}$ (d) none of these

Solution :

(a). $\lim_{x\to 0}\frac{\int_0^{x+y} e^{\sin^2 t}dt - \int_0^{y} e^{\sin^2 t}dt}{x}$

$= \lim_{x\to 0}\frac{\int_y^{0} e^{\sin^2 t}dt + \int_0^{x+y} e^{\sin^2 t}dt}{x}$

$= \lim_{x\to 0}\frac{\int_y^{x+y} e^{\sin^2 t}dt}{x}$

$= \lim_{x\to 0}\frac{e^{\sin^2(x+y)}.\frac{d}{dx}(x+y) - e^{\sin^2 y}.\frac{dy}{dx}}{1}$

$= \lim_{x\to 0}\frac{e^{\sin^2(x+y)}\cdot 1 - e^{\sin^2 y}\cdot 0}{1} = e^{\sin^2 y}$

Practice example

Evaluate $\int_0^a (a^2 + x^2)^{5/2}\, dx$.

Solution :

$$I = \int_0^a (a^2 + x^2)^{5/2}\, dx \qquad \text{Put } x = a\tan\theta \quad \therefore\ dx = a\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} (a^2 + a^2\tan^2\theta)^{5/2} \cdot a\sec^2\theta\, d\theta$$

$$= a^6 \int_0^{\pi/4} \sec^7\theta\, d\theta$$

$$= a^6\left[\left(\frac{\sec^5\theta\tan\theta}{6}\right)_0^{\pi/4} + \frac{5}{6}\int_0^{\pi/4}\sec^5\theta\, d\theta\right]$$

$$= a^6\left[\frac{2\sqrt{2}}{3} + \frac{5}{6}\int_0^{\pi/4}\sec^5\theta\, d\theta\right]$$

$$= a^6\left[\frac{2\sqrt{2}}{3} + \frac{5}{6}\left\{\left(\frac{\sec^3\theta + \tan\theta}{4}\right)_0^{\pi/4} + \frac{3}{4}\int_0^{\pi/4}\sec^3\theta\, d\theta\right\}\right]$$

$$= a^6\left[\frac{2\sqrt{2}}{3} + \frac{5}{6}\left\{\frac{2\sqrt{2}}{4} + \frac{3}{4}\int_0^{\pi/4}\sec^3\theta\, d\theta\right\}\right]$$

$$= a^6\left[\frac{2\sqrt{2}}{3} + \frac{5\sqrt{2}}{12} + \frac{5}{8}\int_0^{\pi/4}\sec^3\theta\, d\theta\right]$$

$$= a^6\left[\frac{2\sqrt{2}}{3} + \frac{5\sqrt{2}}{12} + \frac{5}{8}\left\{\left(\frac{\sec\theta\tan\theta}{2}\right)_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\sec\theta\, d\theta\right\}\right]$$

$$= a^6\left[\frac{2\sqrt{2}}{3} + \frac{5\sqrt{2}}{12} + \frac{5}{8}\left\{\frac{\sqrt{2}}{2} + \frac{1}{2}\{\log(\sec\theta + \tan\theta)\}_0^{\pi/4}\right\}\right]$$

$$= a^6\left[\frac{2\sqrt{2}}{3} + \frac{5\sqrt{2}}{12} + \frac{5\sqrt{2}}{16} + \frac{5}{16}\log(\sqrt{2} + 1)\right]$$

$$= a^6\left[\frac{32\sqrt{2}}{48}+\frac{20\sqrt{2}}{48}+\frac{15\sqrt{2}}{48}+\frac{5}{16}\log\left(\sqrt{2}+1\right)\right]$$

$$= a^6\left[\frac{32\sqrt{2}+20\sqrt{2}+15\sqrt{2}}{48}+\frac{5}{16}\log\left(\sqrt{2}+1\right)\right]$$

$$= a^6\left[\frac{67\sqrt{2}}{48}+\frac{5}{16}\log\left(\sqrt{2}+1\right)\right]$$

$$= \frac{a^6}{48}\left[67\sqrt{2}+15\log\left(\sqrt{2}+1\right)\right]$$

Practice example

$\int_0^5 \frac{\tan^{-1}(x-[x])}{1+(x-[x])^2}\,dx$, where [·] denotes the greatest integer function, is equal to

(a) $\frac{\pi^2}{32}$ (b) $\frac{3\pi^2}{32}$

(c) $\frac{5\pi^2}{32}$ (d) none of these

Solution :

(c). $\int_0^5 \frac{\tan^{-1}(x-[x])}{1+(x-[x])^2}\,dx$

$$= \int_0^5 \frac{\tan^{-1}(x-[x])}{1+(x-[x])^2}\,dx$$

$$= \int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx + \int_1^2 \frac{\tan^{-1}(x-1)}{1+(x-1)^2}\,dx + \ldots$$

$$+\int_4^5 \frac{\tan^{-1}(x-4)}{1+(x-4)^2}\,dx$$

$$=\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx+\int_0^1 \frac{\tan^{-1}t}{1+t^2}\,dt+\ldots+\int_0^1 \frac{\tan^{-1}t}{1+t^2}\,dt$$

(Putting $x-1=t$) (Putting $x-4=t$)

$$=5\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx=5\int_0^{\pi/4} u\,du \quad [\text{Putting } \tan^{-1}x=u]$$

$$=5\left[\frac{u^2}{2}\right]_0^{\pi/2}=\frac{5\pi^2}{32}$$

Practice example

Let $I_1=\int_{\sec^2 z}^{2-\tan^2 z} x\,f(x(3-x))\,dx$

and, $I_2=\int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x))\,dx,$

where f is a continuous function and z is any real number, then $I_1/I_2=$

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$

(c) 1 (d) none of these

Solution

(a). We have, $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\, f(x(3-x))\, dx$

$$= \int_{\sec^2 z}^{2-\tan^2 z} (3-x)\, f((3-x)\{3-(3-x)\})\, dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$$

$$= \int_{\sec^2 z}^{2-\tan^2 z} (3-x)\, f(x(3-x))$$

$$= 3\int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x))\, dx - \int_{\sec^2 z}^{2-\tan^2 z} x f(x(3-x))\, dx$$

$$= 3\, I_2 - I_1$$

$\therefore \quad 2\, I_1 = 3\, I_2$ and so $I_1/I_2 = \dfrac{3}{2}$

Practice example

Evaluate $\int_0^{\pi/4} \tan^5 \theta\, d\theta$

$$I = \int_0^{\pi/4} \tan^5 \theta\, d\theta$$

$$= \left(\frac{\tan^4\theta}{4}\right)_0^{\pi/4} - \int_0^{\pi/4} \tan^3\theta \, d\theta$$

$$= \frac{1}{4} - \int_0^{\pi/4} \tan^3\theta \, d\theta$$

$$= \frac{1}{4} - \left[\left(\frac{\tan^2\theta}{2}\right)_0^{\pi/4} - \int_0^{\pi/4} \tan\theta \, d\theta\right]$$

$$= \frac{1}{4} - \left[\frac{1}{2} - (\log\sec\theta)_0^{\pi/4}\right]$$

$$= \frac{1}{4} - \left[\frac{1}{2} - \log\sqrt{2}\right]$$

$$= -\frac{1}{4} + \log\sqrt{2}$$

$$= -\frac{1}{4} + \frac{1}{2}\log 2$$

Practice example

If $\varphi(n) = \int_0^{\pi/4} \tan^n x \, dx$, show that $\varphi(n) + \varphi(n-2) = \frac{1}{n-1}$ and deduce the value of $\varphi(5)$.

Solution :

$$\varphi(n) = \int_0^{\pi/4} \tan^n x \, dx$$

$$= \left(\frac{\tan^{n-1} x}{n-1}\right)_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1} - \varphi_{n-2}$$

$$\Rightarrow \varphi_n + \varphi_{n-2} = \frac{1}{n-1} \qquad \text{Proved}$$

Now $\varphi(5) = \frac{1}{4} - \varphi_3$

$$= \frac{1}{4} - \left[\frac{1}{2} - \varphi_1\right]$$

$$= -\frac{1}{4} + \varphi_1$$

$$= -\frac{1}{4} + \int_0^{\pi/4} \tan x \, dx$$

Practice Example

Prove that

$$\int_0^{\pi/2} \cos^m x \sin mx \, dx = \frac{1}{2^{m+1}}\left\{2 + \frac{2^2}{2} + \frac{2^3}{3} + \ldots\ldots + \frac{2^m}{m}\right\}$$

Solution :

We know that

$$\int_0^{\pi/2} \cos^m x \sin mx \, dx$$

$$= \left[-\frac{\cos^m x \cos mx}{m+m}\right]_0^{\pi/2} + \frac{m}{m+m} \int_0^{\pi/2} \cos^{m-1} x \sin(m-1)x \, dx$$

$$\Rightarrow \qquad I_{m,m} = \frac{1}{2m} + \frac{1}{2} I_{m-1,m-1}$$

Put $m - 1$ for m,

$$I_{m-1,m-1} = \frac{1}{2(m-1)} + \frac{1}{2} I_{m-2,m-2}$$

$$I_{m,m} = \frac{1}{2m} + \frac{1}{2}\left[\frac{1}{2(m-1)} + \frac{1}{2} I_{m-2,m-2}\right]$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2} I_{m-2,m-2}$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \frac{1}{2^3} I_{m-3,m-3}$$

| Proceeding similarly

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \ldots\ldots + \frac{1}{2^m\{m-(m-1)\}} + \frac{1}{2^m} I_{m-m,m-m}$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \ldots\ldots + \frac{1}{2^m 1} + \frac{1}{2^m} I_{o,o}$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \ldots\ldots + \frac{1}{2^m \cdot 1} + \frac{1}{2^m}\int_0^{\pi/2} o\,dx$$

Now $\int_0^{\pi/2} o\,dx = [c]_0^{\pi/2} = c - c = o$

$$\therefore \quad I_{m,m} = \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \ldots\ldots + \frac{1}{2^m \cdot 1}$$

Writing the series in the reverse order

$$= \frac{1}{2^m \cdot 1} + \frac{1}{2^{m-1} \cdot 2} + \frac{1}{2^{m-2} \cdot 3} + \ldots\ldots + \frac{1}{2m}$$

$$= \frac{1}{2^{m+1}}\left[\frac{2^{m+1}}{2^m \cdot 1} + \frac{2^{m+1}}{2^{m-1} \cdot 2} + \frac{2^{m+1}}{2^{m-2} \cdot 3} + \ldots\ldots + \frac{2^{m+1}}{2m}\right]$$

Practice Example

Prove that $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{n-1}$; n being an integer greater than unity.

Solution :

$$I = \int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx$$

$$= \int_0^{\pi/2} \cos^{n-2} x \sin \{(n-1)x + x\} \, dx$$

$$= \int_0^{\pi/2} \cos^{n-2} x \{\sin (n-1) x \cos x + \cos (n-1) x \sin x\} \, dx$$

$$= \int_0^{\pi/2} \underset{\text{I}}{\cos^{n-1} x} \, \underset{\text{II}}{\sin (n-1) x} \, dx + \int_0^{\pi/2} \cos^{n-2} x \cos (n-1) x \sin x \, dx$$

Integrating the first integral only by parts

$$= \left\{\cos^{n-1} x \cdot -\frac{\cos (n-1) x}{n-1}\right\}_0^{\pi/2} - \int_0^{\pi/2} (n-1) \cos^{n-2} x (-\sin x) \cdot \left\{-\frac{\cos (n-1) x}{n-1}\right\} dx + \int_0^{\pi/2} \cos^{n-2} x \cos (n-1) x \sin x \, dx$$

$$= \frac{1}{n-1} - \int_0^{\pi/2} \cos^{n-2} x \cos (n-1) x \sin x \, dx + \int_0^{\pi/2} \cos^{n-2} x \cos (n-1) x \sin x \, dx$$

$$= \frac{1}{n-1}$$

Practice Example

If $I_{1,n} = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx$ and $I_{2,n} = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$

$n \in N$, then

(a) $I_{2,n+1} - I_{2,n} = I_{1,n}$

(b) $I_{2,n+1} - I_{2,n} = I_{1,n+1}$

(c) $I_{2,n+1} + I_{1,n} = I_{2,n}$

(d) $I_{2,n+1} + I_{1,n+1} = I_{2,n}$

Solution

(b). $I_{2,n} - I_{2,n-1} = \int_0^{\pi/2} \frac{(\sin^2 nx - \sin^2 (n-1)x)}{\sin^2 x} dx$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)x \sin x}{\sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx = I_{1,n}$$

$\therefore \quad I_{2,n+1} - I_{2,n} = I_{1,n+1}$

Reduction forms

Let $I_n = \int \sin^n x\,dx$ or $I_n = \int \sin^{n-1} x \sin x\,dx$.

Integrating by parts regarding sin x as the 2nd function, we have

$I_n = \sin^{n-1} x.(-\cos x) - \int (n-1)\sin^{n-2} x.\cos x.(-\cos x)dx$

$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x.\cos^2 x\,dx$

$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x.(1-\sin^2 x)dx$

$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$

$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)I_n$.

Transposing the last term to the left, we have

$I_n (1 + n - 1) = -\sin^{n-1} x.\cos x + (n-1) I_{n-2}$,

$$\left[\because I_{n-2} = \int \sin^{n-2} x\,dx\right]$$

or $nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

or $I_n = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$.

Let $I_n = \int \cos^n x\,dx$ or $I_n = \int \cos^{n-1} x.\cos x\,dx$.

Integrating by parts regarding cos x as the 2nd function, we have

$I_n = \cos^{n-1} x.\sin x - \int (n-1)\cos^{n-2} x.(\sin x).\sin x\,dx$

$= \cos^{n-1} x.\sin x + (n+1)\int \cos^{n-2} x.\sin^2 x\,dx$

$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$

$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$

$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$.

Transposing the last term to the left, we have

$I_n (1 + n - 1) = \cos^{n-1} x.\sin x + (n-1) I_{n-2}$

or $n I_n = \cos^{n-1} x.\sin x + (n-1) I_{n-2}$.

$\therefore \int \cos^n dx = \dfrac{\cos^{n-1} x \sin x}{n} + \dfrac{n-1}{n}\int \cos^{n-2} x\,dx$.

We have $\int \tan^n x\,dx = \int \tan^{n-2} x.\tan^2 x\,dx$

$= \int \tan^{n-2} x.\left(\sec^2 x - 1\right)dx$

$= \int \tan^{n-2} x.\sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$= \dfrac{(\tan x)^{n-2+1}}{n-2+1} - \int \tan^{n-2} x\,dx$

or $\int \mathbf{\tan^n x\,dx} = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx,$

We have $\int \cot^n x\,dx = \int \cot^{n-2} x.\cot^2 x\,dx$

$= \int \cot^{n-2} x\left(\operatorname{cosec}^2 x - 1\right)dx$

$= \int \cot^{n-2} x.\operatorname{cosec}^2 x\,dx - \int \cot^{n-2} x\,dx$

$= -\dfrac{(\cot x)^{n-1}}{n-1} - \int \cot^{n-2} x\,dx$

or $\cot^n x\,dx = -\dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx,$

We have $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x.\sec^2 x\,dx$

Integrating by parts regarding $\sec^2 x$ as the 2nd function, we have

$I_n = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan^2 x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\,dx + (n+2)\int \sec^{n-2} x\,dx.$

Transposing the term containing $\int \sec^n x\,dx$ to the left,

we have

$(n-2+1)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx$

$$\int \sec^n x \, dx = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan x \tan x \, dx$$

$$= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$= \tan x \sec^{n-2} x - (n-2) \left(\sec^n x - \int \sec^{n-2} x \, dx \right)$$

$$[1 + (n-2)] \int \sec^n x \, dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x \, dx$$

$$\int \sec^n x \, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

- $\int \operatorname{cosec}^n x \, dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$

Integrating by parts,

$$\int \operatorname{cosec}^n x \, dx = \operatorname{cosec}^{n-2} x \, (-\cot x) - \int (n-2) \operatorname{cosec}^{n-3} x \, (-\operatorname{cosec} x \cot x)(-\cot x) \, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2) \int \operatorname{cosec}^{n-2} x \, (\operatorname{cosec}^2 x - 1) \, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2) \left(\int \operatorname{cosec}^n x - \int \operatorname{cosec}^{n-2} x \, dx \right)$$

$$[1 + (n-2)] \int \operatorname{cosec}^n x \, dx = -\cot x \operatorname{cosec}^{n-2} x + (n-2) \int \operatorname{cosec}^{n-2} x \, dx$$

$$\int \operatorname{cosec}^n x \, dx = \frac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} x \, dx$$

-

Definite integral of e to the power (x + 5) square explained and Discussed at

1) https://archive.org/details/1HaveFunWithAbhishekDefiniteIntegralCookedButReasonOfCookingUnknown

2) https://archive.org/details/2TejaDefiniteIntegralCookedWellReasonOfCookingKnownSabjiCutIntoSmallPieces

3) https://archive.org/details/3TejaDefiniteIntegralCookedWellReasonOfCookingKnownSabjiCutIntoSmallPieces

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Various ways of integrating a function d($\tan^{-1}$ 1/x)

https://archive.org/details/DDxOfAFunctionIntegratedInVariousWaysAvoidAnInterestingMistakePart3

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Definite Integral of Sin Square t to 1 + Cos Square t xf(x(2-x)) Discussed and explained at

https://archive.org/details/DefiniteIntegralAToBXIsReplacedWithAPlusBMinusXAndSimplifies

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Definite Integration of Greatest Integer Function 0 to π of [2 Sin x]

https://archive.org/details/DefiniteIntegrationOfGreatestIntegerFunctionIITJEEPart2

IIT-JEE 1995 greatest integer function of [2 Sin x]

https://archive.org/details/DefiniteIntegrationOfGreatestIntegerFunctionIITJEEPart1995

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Differentiation of a Definite Integral Leibniz form discussed and explained at

https://archive.org/details/DifferentiationOfADefiniteIntegralExampleLeibnizFormPart1

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https://archive.org/details/ExtremelyImportantAndRareDefiniteIntegralSolvedByPartialDifferentiation1

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Definite Integral 0 to 1 of 1 + e to the power -x square IIT-JEE 1981 Discussed and explained at

https://archive.org/details/IITJEE1981IntegralCalculusToBeSolvedByExpansionEToThePowerMinusXSquare

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IIT JEE 1990 Definite Integral of odd function (f(x)+f(-x))(g(x)-g(-x)) explained and solved at

https://archive.org/details/IITJEE1990DefiniteIntegralOfOddFunction

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To recall standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
		$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert > a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a > 0)$
		$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2+x^2}}{a^2}\right]$
		$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{x^2-a^2}}{a^2}\right]$

Some series Expansions -

$$\frac{\pi}{2} = \left(\frac{2}{1}\frac{2}{3}\right)\left(\frac{4}{3}\frac{4}{5}\right)\left(\frac{6}{5}\frac{6}{7}\right)\left(\frac{8}{7}\frac{8}{9}\right)\ldots$$

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \cdots$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\pi = \sqrt{12}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \cdots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}$$

Solve a series problem

If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ upto $\infty = \frac{\pi^2}{6}$, then value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ up to ∞ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{6}$ (c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{12}$

Ans. (c)

Solution We have $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ upto ∞

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \cdots \text{ upto } \infty$$

$$- \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$= \frac{\pi^2}{6} - \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots \infty = \frac{\pi^2}{12}$$

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \infty = \frac{\pi^2}{24}$$

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \frac{x^4}{9!} - \frac{x^5}{11!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1 \le x < 1)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} \cdots + \frac{2^{2n}\left(2^{2n}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{E_n x^{2n}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}$$

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots + \frac{2\left(2^{2n-1}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - - \cdots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} - \cdots \quad 0 < |x| < \pi$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\sec x = 1 + \frac{x^2}{\lfloor 2} + \frac{5x^4}{\lfloor 4} + \cdots$$

$$\log(\cos x) = -\frac{x^2}{\lfloor 2} - \frac{2x^4}{\lfloor 4} - \cdots$$

$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \quad |x| < 1$$

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$= \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots\right) \quad |x| < 1$$

$$\tan^{-1} x = \begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & |x| < 1 \\ \pm\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & \begin{cases} + \text{ if } x \geq 1 \\ - \text{ if } x \leq -1 \end{cases} \end{cases}$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$= \frac{\pi}{2} - \left(\frac{1}{x} + \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} + \cdots\right) \quad |x| > 1$$

$$\csc^{-1} x = \sin^{-1}(1/x)$$

$$= \frac{1}{x} + \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} + \cdots \quad |x| > 1$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$= \begin{cases} \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) & |x| < 1 \\ p\pi + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots & \begin{cases} p = 0 \text{ if } x \geq 1 \\ p = 1 \text{ if } x \leq -1 \end{cases} \end{cases}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right]$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{x-1}{x+1}\right)^{2n-1} \quad (x > 0)$$

$$\ln x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x-1}{x}\right)^n \quad (x > \frac{1}{2})$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}(x-1)^n \quad (0 < x \le 2)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n \quad (|x| < 1)$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\infty \ (-1 \le x < 1)$$

$$\log_e(1+x) - \log_e(1-x) =$$

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty\right) (-1 < x < 1)$$

$$\log_e\left(1 + \frac{1}{n}\right) = \log_e \frac{n+1}{n} = 2 \qquad \left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\infty\right]$$

$$\log_e (1 + x) + \log_e (1 - x) = \log_e(1 - x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \ldots\infty\right) (-1 < x < 1)$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \quad = \frac{1}{1\cdot2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$$

Important Results

(i) (a) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$

(b) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1+\tan^n x}$

(c) $\int_0^{\pi/2} \frac{dx}{1+\cot^n x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x} dx$

(d) $\int_0^{\pi/2} \frac{\tan^n x}{\tan^n x + \cot^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{\tan^n x + \cot^n x} dx$

(e) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\sec^n x + \operatorname{cosec}^n x} dx$ where, $n \in R$

(ii) $\int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \frac{\pi}{4}$

(iii) (a) $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$

(b) $\int_0^{\pi/2} \log \tan x \, dx = \int_0^{\pi/2} \log \cot x \, dx = 0$

(c) $\int_0^{\pi/2} \log \sec x \, dx = \int_0^{\pi/2} \log \operatorname{cosec} x \, dx = \frac{\pi}{2} \log 2$

(iv) (a) $\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$

(b) $\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

(c) $\int_0^{\infty} e^{-ax} x^n \, dx = \frac{n!}{a^n+1}$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + C$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + C$$

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Good Luck to you for your Preparations, References, and Exams

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